

# Entire solutions of superlinear problems with indefinite weights and Hardy potentials.

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## Abstract

We provide the structure of regular/singular fast/slow decay radially symmetric solutions for a class of superlinear elliptic equations with an indefinite weight on the nonlinearity  $f(u, r)$ . In particular we are interested in the case where  $f$  is positive in a ball and negative outside, or in the reversed situation. We extend the approach to elliptic equations in presence of Hardy potentials. By the use of Fowler transformation we study the corresponding dynamical systems, presenting the construction of invariant manifolds when the global existence of solutions is not ensured.

**Key Words:** supercritical equations, Hardy potentials, radial solutions, regular/singular ground states, Fowler transformation, invariant manifold, continuity.

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## 1 Introduction

A first purpose of this paper is to study the properties of radial solutions for equations of the form

$$\Delta u + f(u, r) = 0 \quad (\text{L})$$

where  $u : \mathbb{R}^n \rightarrow \mathbb{R}$ ,  $n > 2$ ,  $r = |x|$ , and  $f : \mathbb{R} \times (0, +\infty) \rightarrow \mathbb{R}$  is a differentiable function which is null for  $u = 0$  and super-linear in  $u$ . Since we just deal with radial solutions we will indeed consider the following singular ordinary differential equation

$$u'' + \frac{n-1}{r}u' + f(u, r) = 0, \quad (\text{Lr})$$

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where, abusing the notation, we have set  $u(r) = u(x)$  for  $|x| = r$ , and  $'$  denotes differentiation with respect to  $r$ .

Radial solutions play a key role for (L), since in many cases, e.g.  $k(r) \equiv K > 0$ , positive solutions have to be radial (but also in many situations in which  $k$  is allowed to vary, see e.g. [5, 26, 31]). They are also crucial to determine the threshold between fading and blowing up initial data in the associated parabolic problem, see e.g. [14, 39].

In this article we are mainly interested in classifying positive and nodal solutions when  $f(u, r)$  is negative for  $r$  small and positive for  $r$  large, or in the opposite situation. The prototype for the nonlinearity we are interested in takes the form:

$$f(u, r) = k(r)u|u|^{q-2} \quad (1.1)$$

where  $k$  is a continuous function, which is either negative in a ball and positive outside and  $q > 2^*$ , or we have the reversed sign condition and  $2_* < q < 2^* = \frac{2n}{n-2}$ , where  $2_* := \frac{2(n-1)}{n-2} < 2^* := \frac{2n}{n-2}$ , are respectively the Serrin and the Sobolev critical exponents.

The behavior of solutions of (Lr), with  $f$  as in (1.1), changes drastically according to the sign of  $k$ , when  $q > 2$ . When  $k(r) < 0$  for any  $r > 0$ , positive solutions are convex, and their maximal interval of continuation may be bounded either from above or from below or both. On the other hand if  $k(r) > 0$  all the solutions of (Lr) are continuable for any  $r > 0$ ; further, if  $k(r) > 0$  the structure of positive solutions of (Lr) changes drastically when the exponent  $q$  in (1.1) passes through some critical values, such as  $2_*$  and  $2^*$ , see e.g. [20, 35]. In fact new and more complex situations arise when the non-linearity exhibits both subcritical and supercritical behavior with respect to these exponents, see e.g. [3, 4, 5, 23, 24, 40], for a far from being exhaustive bibliography. In fact we have an interaction between the exponent and the asymptotic behavior of  $k$ . Roughly speaking, if  $f$  is of type (1.1) and  $k(r)$  behaves like a positive power, then the critical exponents get smaller, while they get larger if  $k$  behaves like a negative power. E.g., if  $k(r) = r^\delta$  then the Sobolev critical exponent becomes  $2_\delta^* = 2\frac{n+\delta}{n-2}$ .

With very weak assumptions, definitively positive solutions exhibit two behaviors as  $r \rightarrow 0$  and as  $r \rightarrow \infty$  when  $k(r) > 0$ . Namely  $u(r)$  may be a *regular solution*, i.e.  $u(0) = d > 0$  and  $u'(0) = 0$ , or a *singular solution*, i.e.  $\lim_{r \rightarrow 0} u(r) = +\infty$ ; a *fast decay (f.d.) solution*, i.e.  $\lim_{r \rightarrow \infty} u(r)r^{n-2} = L$ , or a *slow decay (s.d.) solution*, i.e.  $\lim_{r \rightarrow \infty} u(r)r^{n-2} = +\infty$ . We emphasize that in many situations the behavior of singular and slow decay solutions can be specified better (cf. Remark 2.6).

In the whole paper we use the following notation: we denote by  $u(r, d)$  the regular solution of (Lr) such that  $u(0, d) = d$ , and by  $v(r, L)$  the fast decay solution such that  $\lim_{r \rightarrow \infty} r^{n-2}v(r, L) = L$ . Moreover we call *ground states* (G.S.), regular positive solutions  $u(r)$  defined for any  $r \geq 0$  and such that  $\lim_{r \rightarrow \infty} u(r) = 0$ , and *singular ground states* (S.G.S.) positive singular solutions  $v(r)$  defined for any  $r > 0$  and such that  $\lim_{r \rightarrow 0} v(r) = \infty$ ,  $\lim_{r \rightarrow \infty} v(r) = 0$ .

In this paper we continue the discussion, begun with [25], which is mainly

focused on the case where  $f$  is of type (1.1),  $q > 2^*$ ,  $k(r)$  is discontinuous and equals 1 inside a ball and  $-1$  outside. This kind of equation is a special reaction-diffusion equation, where the reaction, modeled by  $f$ , is assumed to have a source effect inside a ball and an absorption effect outside. So it can describe, e.g., the temperature  $u$  in presence of a nonlinear reaction producing energy (taking place in a bounded box) and its inverse absorbing it (taking place in the environment where the box is immersed), both heat regulated. As specified in [25] it can also describe the density of a substance subject to diffusion and to a nonlinear reaction and its inverse, see also [34, §7]. The inhomogeneity may be induced by the presence of an activator or an inhibitor.

In [25] the purpose was to prove existence and exact multiplicity for regular solutions with f.d. and with s.d., and to deduce their nodal properties, but just for a very specific example, discontinuous in  $r$ .

Here we want to show that the case described in [25] is the prototype for a large class of nonlinearities  $f$ . So we relax the requirement on  $k(r)$ ; in particular we assume it to be smooth, and we extend the results to a wider family of potentials  $f$ , whose main representative is given by

$$\begin{aligned} f(u, r) &= K(r)u|u|^{q-2} & q > 2^* & \quad (a) \\ f(u, r) &= -K(r)u|u|^{q-2} & 2_* < q < 2^* & \quad (b) \end{aligned} \tag{1.2}$$

where  $K(r)$  changes sign one time: in particular  $K(r) < 0$  if  $r < R$ ,  $K(r) > 0$  if  $r > R$ , for a certain  $R$ . Section 4 will be devoted to a deeper analysis of the possible nonlinearities we can deal with, but we wish to emphasize that our result is new for this kind of nonlinearity, too.

We emphasize that the presence of G.S. with f.d. is due to the coexistence of source and absorption effects (i.e.  $f$  changes sign). In literature there are many results on the structure of radial solutions for Laplace equations with indefinite weights  $k$ , see e.g. [1, 4, 7]. However, these papers are concerned with phenomena which are found when  $k$  is a positive function, and which persist even if  $k$  becomes negative in some regions. The structure results we find can just take place if we have a change in the sign of  $k$ : if  $q$  is either smaller or larger than  $2^*$  there are no G.S. with fast decay, neither if  $k(r) \equiv K > 0$ , nor if  $k(r) \equiv K < 0$ . In fact, the structure of the solutions of (Lr) described in Corollaries 1.2 and 1.3, reminds of the situation in which  $q = 2^*$  and we have a positive  $k$  which behaves like a positive power for  $r$  small and a negative power for  $r$  large, see e.g. [13, 40]. In the same direction goes [6], which proves existence results (using a variational approach) which hold just when the nonlinearities have sign-changing weights; however in [6] the authors consider bounded domains and just in the subcritical case. Further in [15, 32, 33] and in references therein the reader can find several nice and sharp structure results for sign-changing nonlinearities, even for more general operators ( $p$ -Laplace, relativistic and mean curvature), in the framework of oscillation (and non-oscillation) theory, but for exterior domains, i.e. for solutions defined, say for  $r > 1$ .

Performing this generalization with respect to [25] we pay two prizes: firstly we can just give existence and multiplicity results, but we lose the control of

uniqueness and exact multiplicity since we mainly ask for asymptotic conditions; secondly we have to face many technical problems and the discussion is more involved. The major one is the following: asking for  $f$  to be negative and possibly superlinear for  $u$  large, we allow the existence of non-continuable solutions, whose presence causes almost no difficulties for the special nonlinearities considered in [25], but it is a crucial problem here.

In fact the presence of non-continuable solutions, which are typical for the nonlinearity considered, rises a challenging problem from the theoretical point of view. In fact we cannot apply the already established invariant manifold theory for non-autonomous systems (see e.g. [10, 12, 27]). In the appendix we perform a first step in order to extend this theory to the case where non-continuable solutions are allowed. As far as we are aware this is the first time where such a problem is considered, and we think this can be a contribution from a methodological point of view to invariant manifold theory for non-autonomous dynamical systems.

Our analysis is directly performed for the following more general differential equation

$$\Delta u + \frac{h(r)}{r^2}u + f(u, r) = 0, \quad (\text{H})$$

and for its radial counterpart

$$u'' + \frac{n-1}{r}u' + \frac{h(r)}{r^2}u + f(u, r) = 0. \quad (\text{Hr})$$

We assume that  $h$  is a differentiable function satisfying the following requirement, which will be assumed in the whole paper:

$$\begin{aligned} \text{H} \quad & h(r) < \frac{(n-2)^2}{4}, \quad \text{for every } r \in (0, +\infty), \\ & h(0) = \eta < \frac{(n-2)^2}{4}, \quad \frac{h(r)-\eta}{r} \in L^1(0, 1], \\ & \lim_{r \rightarrow \infty} h(r) = \beta < \frac{(n-2)^2}{4}, \quad \frac{h(r)-\beta}{r} \in L^1[1, +\infty). \end{aligned}$$

The introduction in Laplace equation of the additional term  $\frac{h(r)}{r^2}u$ , often referred to as *Hardy potential*, has raised a great interest recently, see e.g. [2, 18, 38], and we think is another main point of interest in our paper. Usually in literature the case  $h \equiv \eta$  is considered, with the requirement that  $\eta \leq \eta_c := \frac{(n-2)^2}{4}$  (the value  $\eta_c$  is again critical). The restriction  $\eta \leq \eta_c$  is necessary in order to have definitively positive solutions, see e.g. [9], either for  $r$  small or for  $r$  large, and  $\eta_c$  can be interpreted as the first eigenvalue of the  $\Delta u + u/r^2$ , see Section 1 in [38]. Here we give a dynamical interpretation of this assumption. Equation (Hr) has been subject to deep investigation for different type of  $f$ , see e.g. [2, 16, 17, 18, 38]. Usually  $h$  is assumed to be a constant, and there are very few results concerning the case where  $h$  actually varies; however Terracini in [38] and Felli et al. in [16] considered the case where  $h$  is a function, depending in fact on its angular coordinates (to model a magnetic field).

A consequence of the presence of the Hardy term is a shift on the Serrin critical exponent, and the appearance of a new critical value in the supercritical

regime. More precisely, if  $h(r) \equiv \eta < \eta_c = \frac{(n-2)^2}{4}$  we define

$$2_*(\eta) := 2 \frac{n + \sqrt{(n-2)^2 - 4\eta}}{n-2 + \sqrt{(n-2)^2 - 4\eta}} \quad (1.3)$$

(which gives back  $2_*$  if  $\eta = 0$ ), and

$$I(\eta) := \begin{cases} +\infty & \text{if } \eta \leq 0 \\ 2 \frac{n - \sqrt{(n-2)^2 - 4\eta}}{n-2 - \sqrt{(n-2)^2 - 4\eta}} & \text{if } 0 < \eta < \frac{(n-2)^2}{4}, \end{cases} \quad (1.4)$$

see, e.g. [9]. Notice that  $\lim_{\eta \rightarrow \eta_c} 2_*(\eta) = 2^* = \lim_{\eta \rightarrow \eta_c} I(\eta)$ .

The presence of the Hardy term, affects greatly the asymptotic behavior of the solution. In fact if  $f(u, r) > 0$  we continue to have two possible behavior for definitively positive solutions either for  $r$  small or for  $r$  large. Let us set

$$\kappa(\eta) := \frac{(n-2) - \sqrt{(n-2)^2 - 4\eta}}{2}; \quad (1.5)$$

we introduce the following terminology.

**Definition 1.1.** • A  $\mathcal{R}$ -solution  $u(r, d)$  satisfies  $\lim_{r \rightarrow 0} u(r, d)r^{\kappa(\eta)} = d \in \mathbb{R}$ , while a  $\mathcal{S}$ -solution  $u$  satisfies  $\lim_{r \rightarrow 0} u(r)r^{\kappa(\eta)} = \pm\infty$ .

- A fd-solution  $v(r, L)$  satisfies  $\lim_{r \rightarrow \infty} v(r, L)r^{n-2-\kappa(\eta)} = L \in \mathbb{R}$ , while a sd-solution  $u$  satisfies  $\lim_{r \rightarrow \infty} u(r)r^{n-2-\kappa(\eta)} = \pm\infty$ .
- a  $\mathcal{R}^k$  fd solution  $u(r, d) = v(r, L)$  is both a  $\mathcal{R}$ -solution and a fd-solution having  $k$  nondegenerate zeros. We define similarly  $\mathcal{R}^k$  sd solution  $u(r, d)$ ,  $\mathcal{S}^k$  fd solution  $v(r, L)$ . When we do not indicate the value  $k$ , e.g.  $\mathcal{S}$ -fd, we mean any solution with these asymptotic properties disregarding its number of zeroes.

In case of equation (Lr) we can recognize respectively regular fast decay, regular slow decay and singular fast decay solutions having  $k$  nondegenerate zeros.

Note that  $0 < \kappa(\eta) < \frac{n-2}{2}$  if  $\eta > 0$  and that  $\kappa(\eta) < 0$  if  $\eta < 0$ , therefore  $\mathcal{R}$ -solutions are unbounded if  $\eta > 0$ . We also emphasize that bounded solutions do not exist for  $\eta > 0$ , and that  $\mathcal{S}$ -solutions are anyway larger than  $\mathcal{R}$ -solutions for  $r$  small, see Remarks 2.7, 2.8, 2.10 for more details.

This fact may cause relevant problems in applying variational or functional techniques, but in fact it finds an easy explanation with our approach. However the structure of positive solutions is not greatly altered by the presence of the Hardy potential so that we can give here a unified approach for both (L) and (H).

Consider a function  $f$  as in (1.2); we state the following assumption for  $K$ :

**K** Assume that  $K$  is  $C^1$  and there is  $R > 0$  such that  $K(r) < 0$  for  $0 < r < R$  and  $K(r) > 0$  for  $r > R$ . Further assume that

$$\begin{aligned} K(r) &= K(0)r^{\delta_0} + o(r^{\delta_0}) & \text{as } r \rightarrow 0, & \quad \text{and} \\ K(r) &= K(\infty)r^{\delta_\infty} + o(r^{\delta_\infty}) & \text{as } r \rightarrow \infty. \end{aligned} \quad (1.6)$$

where  $K(0) < 0 < K(\infty)$  and  $\delta_0, \delta_\infty > -2$ . Further there is  $\varpi > 0$  (small) such that  $\lim_{r \rightarrow 0} r^{-\varpi} \frac{d}{dr} [K(r)r^{-\delta_0}] = 0$ , and  $\lim_{r \rightarrow \infty} r^{\varpi} \frac{d}{dr} [K(r)r^{-\delta_\infty}] = 0$ .

Note that the weak assumption on the derivative of  $K$  is just technical. We need to introduce the following parameters which take into account of the shift on the critical exponent due to the presence of the spatial dependence:

$$l = l(q, \delta) = 2 \frac{q + \delta}{2 + \delta} \quad (1.7)$$

and notice that  $l(q, 0) = q$ .

We postpone the statement of our main results in a more general framework to Section 2.6. We just propose here two corollaries which follow directly from Theorems 2.15 and 2.16 and apply to nonlinearities introduced in (1.2).

**Corollary 1.2.** *Assume **H**, let  $f$  be as in (1.2a) and suppose  $K(r)$  satisfies **K**. Set  $l_u = l(q, \delta_0)$  and  $l_s = l(q, \delta_\infty)$ , and assume  $2_*(\eta) < l_u < I(\eta)$ ,  $2^* < l_s < I(\beta)$ . Then there is an increasing sequence  $(A_k)_{k \geq 0}$  such that  $u(r, A_k)$  is a  $\mathcal{R}^{\frac{1}{2}} \text{fd}$ . Moreover  $u(r, d)$  is a  $\mathcal{R}^0 \text{sd}$  for any  $0 < d < A_0$ , and there is  $A_k^* \in [A_{k-1}, A_k]$  such that  $u(r, d)$  is a  $\mathcal{R}^{\frac{1}{2}} \text{sd}$  whenever  $A_k^* < d < A_k$ , for any  $k \geq 1$ .*

**Corollary 1.3.** *Assume **H**, let  $f$  be as in (1.2b) and suppose  $K(r)$  satisfies **K**. Set  $l_u = l(q, \delta_0)$  and  $l_s = l(q, \delta_\infty)$ , and assume  $2_*(\eta) < l_u < 2^*$ ,  $2_*(\beta) < l_s < I(\beta)$ . Then there is an increasing sequence  $(B_k)_{k \geq 0}$  such that  $v(r, B_k)$  is a  $\mathcal{R}^{\frac{1}{2}} \text{fd}$ . Moreover,  $v(r, L)$  is a  $\mathcal{S}^0 \text{fd}$  for any  $0 < L < B_0$ , and there is  $B_k^* \in [B_{k-1}, B_k]$  such that  $v(r, L)$  is a  $\mathcal{S}^{\frac{1}{2}} \text{fd}$  whenever  $B_k^* < L < B_k$ , for any  $k \geq 1$ . Consequently, there is an increasing sequence  $(A_k)_{k \geq 0}$  such that  $u(r, A_k)$  is a  $\mathcal{R}^{\frac{1}{2}} \text{fd}$  for any  $k \geq 0$ .*

Notice that both the corollaries provide the existence of a positive G.S. with fast decay (the  $\mathcal{R}^0 \text{fd}$ ) for equation (L). The first one gives also existence of positive G.S. with slow decay (the  $\mathcal{R}^0 \text{sd}$ 's) and the second the existence of positive S.G.S. with fast decay (the  $\mathcal{S}^0 \text{fd}$ 's).

The previous corollaries focus on solutions which are *positive near zero*; by the way similar statements hold for *negative near zero* solutions.

In the proofs we apply the classical Fowler transformation, to pass from (Hr) to a system, and we apply phase plane analysis and techniques from the theory of invariant manifold for non-autonomous systems, following the way paved by [28, 29, 30]. Therefore the existence of  $\mathcal{R}^{\frac{1}{2}} \text{fd}$  corresponds to the existence of homoclinic orbit in the introduced dynamical system. The presence of the Hardy potential forces us to abandon the classical results established in [10], and to add a discussion of exponential dichotomy tools, based on [11, 12, 27].

Kelvin inversion  $u(r) \mapsto u(1/r)r^{2-n}$  assumes a particularly clear form when it is combined with Fowler transformation (see Section 2.5). To the best of our knowledge this simple but useful remark appeared for the first time in [23]; here we explore this fact a bit further.

The paper is structured as follows: in Section 2 we introduce Fowler transformation (§2.1), and we explain some well known correspondences between

the new system and the original problem, in the (Lr) case (§2.2), in the (Hr) case (§2.3); then we state our results in the general framework (§2.6). In Section 3.1 we develop some geometrical consideration, which will be actually used in Section 3.2 to prove our main theorems, following some ideas introduced in [3, 13, 30]. In Section 4 we give some further examples of application of our results. In Appendix A.1 we recall some well known facts concerning invariant manifold theory for non-autonomous system, and we explain our extension to a setting where continuability is lost. Appendix A.2 and A.3 are devoted to adapt to our setting some topological ideas already used respectively in [3, 13], and in [22].

## 2 Preliminaries and stating of the results.

### 2.1 Fowler transformation.

We consider equation (Hr), which corresponds to radial solutions of (H). Once we have fixed a constant  $l > 2$ , and the values

$$\alpha_l = \frac{2}{l-2}, \quad \gamma_l = \alpha_l + 2 - n,$$

setting

$$\begin{cases} x_l(t) = u(r)r^{\alpha_l} \\ y_l(t) = u'(r)r^{\alpha_l+1} \end{cases} \quad \text{where } r = e^t, \quad (2.1)$$

we pass from (Hr) to the following

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 \\ -h(e^t) & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} 0 \\ -g_l(x_l, t) \end{pmatrix}, \quad (S)$$

where

$$g_l(x, t) = f(xe^{-\alpha_l t}, e^t)e^{(\alpha_l+2)t}. \quad (2.2)$$

In particular, in the classical Laplace case, i.e. when  $h(t) \equiv 0$ , we find

$$\begin{pmatrix} \dot{x}_l \\ \dot{y}_l \end{pmatrix} = \begin{pmatrix} \alpha_l & 1 \\ 0 & \gamma_l \end{pmatrix} \begin{pmatrix} x_l \\ y_l \end{pmatrix} + \begin{pmatrix} 0 \\ -g_l(x_l, t) \end{pmatrix}. \quad (S_0)$$

The main advantage in this change of variables is that, when  $f$  is of type (1.1), setting  $l = q$  we obtain a system which is not anymore singular. Moreover, if  $h$  and  $k$  are constants, then (S) is autonomous: in fact (2.1) is a slight modification of the original transformation introduced by Fowler [19].

More in general, whenever  $h$  is a constant and  $k(r) = Kr^\delta$ , where  $\delta > -2$  we can set  $l = l(q, \delta) = 2\frac{q+\delta}{2+\delta}$  to get  $g_l(x, t) = Kx_l|x_l|^{q-2}$ , so that (S) is an autonomous system.

Assume first that  $h \equiv 0$ , and  $g_l(x, t) = Kx_l|x_l|^{q-2}$ . In these cases, whenever  $l > 2_*$  and  $K > 0$ , the origin is a saddle and admits a 1-dimensional unstable

manifold  $M^u$  and a 1-dimensional stable manifold  $M^s$ . Moreover we have two critical points  $\mathbf{P}^+ = (P_x, P_y)$  and  $\mathbf{P}^- = (-P_x, -P_y)$  with  $P_x > 0$ , which are stable for  $l > 2^*$ , centers for  $l = 2^*$  and unstable for  $2_* < l < 2^*$ . Using the translation for this context of the Pohozaev identity, see e.g. [37], we can easily draw the phase portrait, in such an autonomous case (a detailed proof in the  $p$ -Laplace context is given in [20], see also [23, 25]).

If  $K < 0$  and  $l > 2_*$  the origin is the unique critical point and both  $M^u$  and  $M^s$  are unbounded curves, see Figure 1.

In fact this analysis is easily extended to any autonomous system  $(S_0)$  satisfying the following assumption, see [13] for details:

**GA** There is  $l > 2_*$  such that  $g_l(x, t) \equiv K g_l(x)$  is  $t$ -independent,  $K \neq 0$  is a constant, and  $g_l(x)/x$  is a function, which is positive increasing for  $x > 0$  and positive decreasing for  $x < 0$ , satisfying

$$\lim_{x \rightarrow 0} \frac{g_l(x)}{x} = 0 \quad \text{and} \quad \lim_{|x| \rightarrow +\infty} \frac{g_l(x)}{x} = +\infty.$$

This way we can consider e.g.  $g_l(x) = k_1 x |x|^{q_1-2} + k_2 x |x|^{q_2-2}$  (i.e.  $f(u, r) = k_1 u |u|^{q_1-2} + k_2 r^\delta u |u|^{q_2-2}$  where  $\delta = \frac{2(q_2-q_1)}{q_1-2}$ ), or  $g_l(x) = k_1 x |x|^{q-2} \ln(|x|)$  (i.e.  $f(u, r) = k_1 u |u|^{q-2} \ln(ur^{\frac{2}{q-2}})$ ) where  $k_1$  and  $k_2$  are positive constants and  $q_1$  and  $q_2$  are larger than 2. So we can consider slightly more general functions  $f$ .

Now we introduce a further notation which will be in force in the whole paper: we denote by  $\mathbf{x}_l(t, \tau, \mathbf{Q}) = (x_l(t, \tau, \mathbf{Q}), y_l(t, \tau, \mathbf{Q}))$  the trajectory of (S) – or  $(S_0)$  – which is in  $\mathbf{Q}$  for  $t = \tau$ .

The following remark underlines the relations between the behaviour of solutions of  $(S_0)$  and of (Lr).

**Remark 2.1.** Assume **GA**. Consider the trajectory  $\mathbf{x}_l(t, \tau, \mathbf{Q})$  of  $(S_0)$  and let  $u(r)$  be the corresponding solution of (Lr); then  $u(r)$  is a regular solution if and only if  $\mathbf{Q} \in M^u$ , while it has fast decay if and only if  $\mathbf{Q} \in M^s$ .

This result can be easily proved using standard tools in invariant manifold theory, see e.g. [20, 23, 25]; we will prove it as a special case of a more general result, Lemma 2.5 below, in the non-autonomous context.

Assume now  $h(t) \equiv \eta < \frac{(n-2)^2}{4}$ : the linearization of system (S) in the origin has real and distinct eigenvalues. Moreover the origin is a saddle iff  $|\alpha_l \gamma_l| > \eta$  or equivalently

$$2_*(\eta) < l < I(\eta), \quad (2.3)$$

where  $2_*(\eta)$  and  $I(\eta)$  have been defined in (1.3), (1.4). The eigenvalues are  $\lambda_1 = \gamma_l + \kappa(\eta) < 0 < \lambda_2 = \alpha_l - \kappa(\eta)$ , where  $\kappa(\eta)$  was defined in (1.5). Let us assume first that  $g_l(x, t) = K g_l(x)$  satisfies **GA** with  $K > 0$ , where the condition  $l > 2_*$  is replaced by  $l > 2_*(\eta)$ . It is straightforward to check that, when the parameters are in the range (2.3), we have again a unique critical point  $\mathbf{P}^+ = (P_x, P_y)$  in  $x > 0$ ; in particular if  $g_l(x) = x|x|^{q-2}$ , we find  $P_x =$



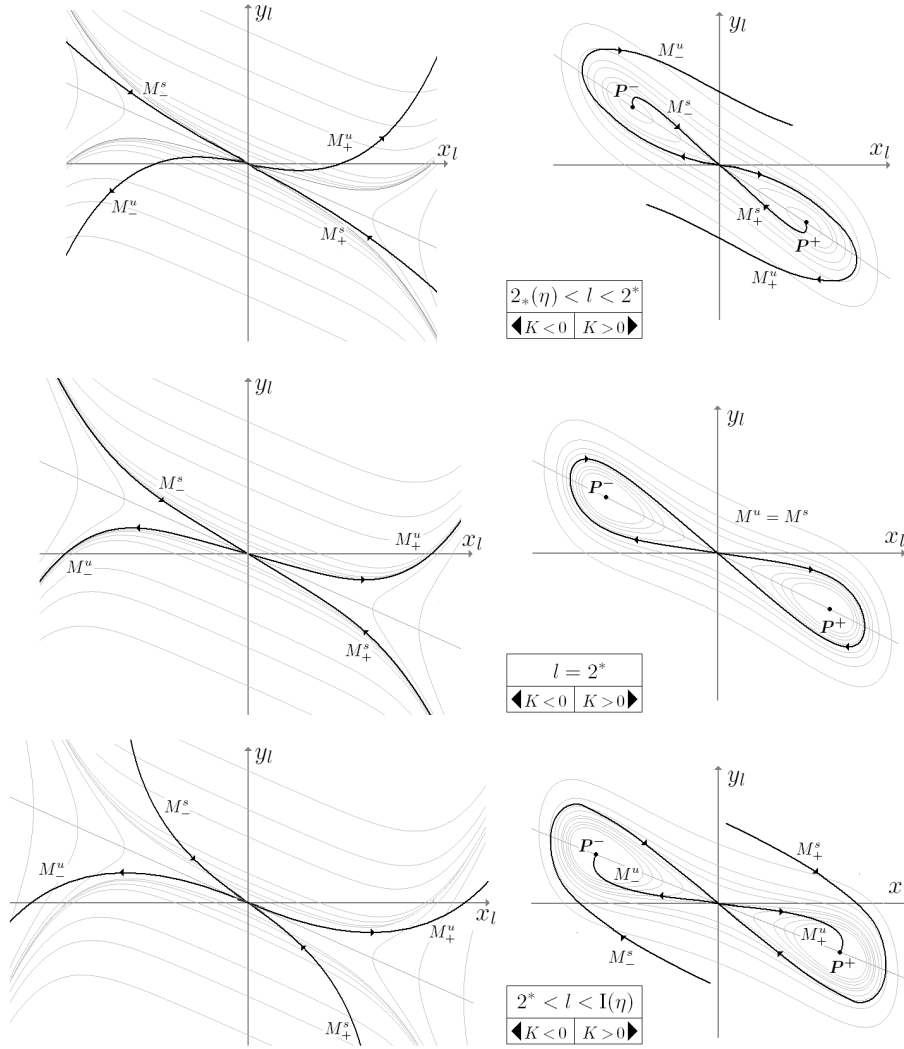


Figure 1: Sketch of the phase portrait of (S), when  $g_l(x, t)$  is  $t$ -independent and satisfies **GA**. The unstable manifolds  $M^u$  and the stable manifolds  $M^s$  are drawn. The manifolds can be located using the level curves of some energy functions (cf. [20, 25, 37]): in particular in the case  $l = 2^*$  the system is Hamiltonian, moreover, if  $K > 0$ , it presents periodic solutions and  $M^u$  and  $M^s$  coincide giving two homoclinic orbits.

$[\alpha_l(n - 2 - \alpha_l) - \eta]^{1/(q-2)}$ , and  $P_y = -\alpha_l P_x$ . The point  $P^+$  is unstable (either a node or a focus) if  $2_*(\eta) < l < 2^*$ , a center if  $l = 2^*$  it is stable if  $2^* < l < I(\eta)$  (either a node or a focus). Again for  $K < 0$  we find that  $M^u$  and  $M^s$  are unbounded curves. See Figure 1. We refer to [25] for details.

We can again consider the stable manifold  $M^s$  and the unstable manifold  $M^u$ , in order to obtain estimates as in Remark 2.1, however the presence of the Hardy potential may forbid the existence of regular solutions. We will present such details in Section 2.3 in the non-autonomous case so to avoid repetitions.

## 2.2 Stable and unstable manifolds for non-autonomous systems.

In the previous subsection we have begun from the autonomous case for illustrative purposes. Now we turn to consider the  $t$ -dependent case: the first step is the generalization of Remark 2.1. In this subsection we will make use of the following assumption for illustrative purposes (it will be removed from the next subsection):

**C** All the trajectories of (S) are continuable for any  $t \in \mathbb{R}$ .

We have two different alternatives to introduce stable and unstable sets for non-autonomous systems, thus extending Remark 2.1 to a generic  $g_l(x, t)$ . The simplest one requires the strongest hypotheses, but gives more structure.

**Gu** Assume **H** and that there is  $l_u \in (2_*(\eta), I(\eta))$  such that  $g_{l_u}(0, t) = 0$ ,  $\partial_x g_{l_u}(0, t) = 0$  for any  $t \in \mathbb{R}$ , and

$$\lim_{t \rightarrow -\infty} g_{l_u}(x, t) = K g_{l_u}^{-\infty}(x) \quad \text{and} \quad \lim_{t \rightarrow -\infty} e^{-\varpi t} \partial_t g_{l_u}(x, t) = 0,$$

uniformly on compact sets, where the function  $g_{l_u}^{-\infty}$  is a non-trivial locally Lipschitz function satisfying **GA** and  $\varpi$  is a suitable positive constant.

**Gs** Assume **H** and that there is  $l_s \in (2_*(\beta), I(\beta))$  such that  $g_{l_s}(0, t) = 0$ ,  $\partial_x g_{l_s}(0, t) = 0$  for any  $t \in \mathbb{R}$ , and

$$\lim_{t \rightarrow +\infty} g_{l_s}(x, t) = K g_{l_s}^{+\infty}(x) \quad \text{and} \quad \lim_{t \rightarrow +\infty} e^{\varpi t} \partial_t g_{l_s}(x, t) = 0,$$

uniformly on compact sets, where the function  $g_{l_s}^{+\infty}$  is a non-trivial locally Lipschitz function satisfying **GA** and  $\varpi$  is a suitable positive constant.

Assume **Gu** and add to (S) the variable  $z = e^{\varpi t}$ , to get

$$\begin{pmatrix} \dot{x}_{l_u} \\ \dot{y}_{l_u} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \alpha_{l_u} & 1 & 0 \\ -h(z^{1/\varpi}) & \gamma_{l_u} & 0 \\ 0 & 0 & \varpi \end{pmatrix} \begin{pmatrix} x_{l_u} \\ y_{l_u} \\ z \end{pmatrix} - \begin{pmatrix} 0 \\ g_{l_u}(x_{l_u}, \ln(z)/\varpi) \\ 0 \end{pmatrix}. \quad (2.4)$$

We have thus obtained an autonomous system and all its trajectories converge to the  $z = 0$  plane as  $t \rightarrow -\infty$ ; so (2.4) is useful to investigate the asymptotic behavior in the past. The origin admits a 2-dimensional unstable manifold denoted by  $\mathbf{W}^u$ . From standard arguments of dynamical system theory, we see

that the set  $W_{l_u}^u(\tau) = \mathbf{W}^u \cap \{z = e^{\varpi\tau}\}$  is a 1-dimensional (immersed) manifold, for any  $\tau \in \mathbb{R}$ , see e.g. [3, 24].

Similarly when  $\mathbf{G}s$  holds we consider the following system to study the behavior of trajectories in the future, adding the new variable  $\zeta = e^{-\varpi t}$ .

$$\begin{pmatrix} \dot{x}_{l_s} \\ \dot{y}_{l_s} \\ \dot{\zeta} \end{pmatrix} = \begin{pmatrix} \alpha_{l_s} & 1 & 0 \\ -h(\zeta^{-1/\varpi}) & \gamma_{l_s} & 0 \\ 0 & 0 & -\varpi \end{pmatrix} \begin{pmatrix} x_{l_s} \\ y_{l_s} \\ \zeta \end{pmatrix} - \begin{pmatrix} 0 \\ g_{l_s}(x_{l_s}, -\ln(\zeta)/\varpi) \\ 0 \end{pmatrix}. \quad (2.5)$$

All its trajectories converge to the  $\zeta = 0$  plane as  $t \rightarrow +\infty$ . The origin admits a 2-dimensional stable manifold denoted by  $\mathbf{W}^s$ . Arguing as above, for any  $\tau \in \mathbb{R}$ ,  $W_{l_s}^s(\tau) = \mathbf{W}^s \cap \{\zeta = e^{-\varpi\tau}\}$  is a 1-dimensional manifold.

Let us denote by  $W_{l_u}^u(-\infty)$  the unstable manifold  $M^u$  of the autonomous system (S) where  $l = l_u$  and  $g_{l_u}(x, t) \equiv K g_{l_u}^{-\infty}(x)$ , and by  $W_{l_s}^s(+\infty)$  the stable manifold  $M^s$  of the autonomous system (S) where  $l = l_s$  and  $g_{l_s}(x, t) \equiv K g_{l_s}^{+\infty}(x)$ . Then we have the following.

**Remark 2.2.** Assume  $\mathbf{C}$  and  $\mathbf{G}u$ ; then  $W_{l_u}^u(\tau)$  approaches  $W_{l_u}^u(-\infty)$  as  $\tau \rightarrow -\infty$ . Assume  $\mathbf{G}s$ ; then  $W_{l_s}^s(\tau)$  approaches  $W_{l_s}^s(+\infty)$  as  $\tau \rightarrow +\infty$ . More precisely, if  $W_{l_u}^u(\tau_0)$  (respectively  $W_{l_s}^s(\tau_0)$ ) intersects transversally a certain line  $L$  in a point  $\mathbf{Q}(\tau_0)$  for  $\tau_0 \in [-\infty, +\infty)$  (resp. for  $\tau_0 \in (-\infty, +\infty]$ ), then there is a neighborhood  $I$  of  $\tau_0$  such that  $W_{l_u}^u(\tau)$  (resp.  $W_{l_s}^s(\tau)$ ) intersects  $L$  in a point  $\mathbf{Q}(\tau)$  for any  $\tau \in I$ , and  $\mathbf{Q}(\tau)$  is continuous (in particular it is as smooth as  $g_l$ ).

The proof of this Remark follows from standard facts in dynamical system theory. If we assume  $h(t) \equiv \eta$ , it follows from [10, § 13], while if we allow  $h$  to be a function satisfying  $\mathbf{H}$  it follows from [12, Theorem 4.1] or [27, Theorem 2.16].

Following [29], which is based on [27], we can introduce stable and unstable leaves with assumptions weaker than  $\mathbf{G}u$  and  $\mathbf{G}s$ , see also the Appendix A.1 for a more detailed discussion of the topic.

**gu** Assume  $\mathbf{H}$  and that there exists  $l_u \in (2_*(\eta), \mathbf{I}(\eta))$  such that  $g_{l_u}(x, t)$  is continuous in  $x$  uniformly for  $t \leq \tau$ , whenever  $\tau \in \mathbb{R}$  and for any  $x$  in a compact set; further  $g_{l_u}(0, t) = \partial_x g_{l_u}(0, t) = 0$  for any  $t \in \mathbb{R}$ .

**gs** Assume  $\mathbf{H}$  and that there exists  $l_s \in (2_*(\beta), \mathbf{I}(\beta))$  such that  $g_{l_s}(x, t)$  is continuous in  $x$  uniformly for  $t \geq \tau$ , whenever  $\tau \in \mathbb{R}$  and for any  $x$  in a compact set; further  $g_{l_s}(0, t) = \partial_x g_{l_s}(0, t) = 0$  for any  $t \in \mathbb{R}$ .

Replacing  $\mathbf{G}u$  by **gu** and  $\mathbf{G}s$  by **gs** we can again construct 1-dimensional (immersed) manifolds  $W_{l_u}^u(\tau)$ , respectively  $W_{l_s}^s(\tau)$ , for any  $\tau \in \mathbb{R}$  by characterizing them as follows:

$$\begin{aligned} W_{l_u}^u(\tau) &:= \{\mathbf{Q} \in \mathbb{R}^2 \mid \lim_{t \rightarrow -\infty} \mathbf{x}_{l_u}(t, \tau, \mathbf{Q}) = (0, 0)\}, \\ W_{l_s}^s(\tau) &:= \{\mathbf{Q} \in \mathbb{R}^2 \mid \lim_{t \rightarrow +\infty} \mathbf{x}_{l_s}(t, \tau, \mathbf{Q}) = (0, 0)\}. \end{aligned} \quad (2.6)$$

Furthermore  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$  in the origin are tangent respectively to the unstable and the stable space of the linearized system, see [27] and the Appendix for more details, in particular Remarks A.2 and A.3.

$W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$  have the smoothness property described above in Remark 2.2, but the first part of Remark 2.2 concerning their asymptotical behavior does not hold anymore (since  $W_{l_u}^u(-\infty)$  and  $W_{l_s}^s(+\infty)$  may be not defined). We stress that  $\mathbf{Gu}$  implies  $\mathbf{gu}$ , and if the former holds then the manifolds  $W_{l_u}^u(\tau)$  constructed via  $\mathbf{Gu}$  and  $\mathbf{gu}$  coincide; the specular result holds for  $\mathbf{Gs}$  which implies  $\mathbf{gs}$  (this way we see that Remark 2.4 below holds for  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$  if we assume  $\mathbf{Gu}$  and  $\mathbf{Gs}$ ). However observe that with  $\mathbf{gu}$  and  $\mathbf{gs}$  we allow also functions  $g_l(x, t)$  which are periodic in  $t$  or which have logarithmic behavior. Further notice that when  $\mathbf{Gu}$  holds, the phase portrait is very different in the two cases  $K > 0$  and  $K < 0$  (see Figure 1), but in any case  $\mathbf{gu}$  holds and guarantees the existence of the unstable manifold. A similar argument holds for  $\mathbf{Gs}$  and  $\mathbf{gs}$ , too.

Since we want to understand the mutual position of these two objects we introduce the manifolds:

$$\begin{aligned} W_{l_s}^u(\tau) &:= \{\mathbf{R} = \mathbf{Q} \exp[-(\alpha_{l_u} - \alpha_{l_s})\tau] \in \mathbb{R}^2 \mid \mathbf{Q} \in W_{l_u}^u(\tau)\}, \\ W_{l_u}^s(\tau) &:= \{\mathbf{Q} = \mathbf{R} \exp[(\alpha_{l_u} - \alpha_{l_s})\tau] \in \mathbb{R}^2 \mid \mathbf{R} \in W_{l_s}^s(\tau)\}. \end{aligned} \quad (2.7)$$

Notice that  $W_{l_s}^u(\tau)$  is omothetic to  $W_{l_u}^u(\tau)$ , and  $W_{l_u}^s(\tau)$  is omothetic to  $W_{l_s}^s(\tau)$ . Hence, they are 1-dimensional (immersed) manifolds for any  $\tau \in \mathbb{R}$ .

**Remark 2.3.** *Observe that when  $\mathbf{gu}$  holds for some  $\bar{l}_u > 2_*(\eta)$ , respectively  $\mathbf{gs}$  holds for some  $\bar{l}_s > 2_*(\beta)$ , then  $\mathbf{gu}$  holds for any  $l_u \in [\bar{l}_u, I(\eta))$ , resp.  $\mathbf{gs}$  holds for any  $l_s \in (2_*(\beta), \bar{l}_s]$ .*

The validity of the previous remark can be immediately verified: if we choose  $l \neq L$  we have  $g_L(x, t)/x = g_l(\xi, t)/\xi$ , where  $\xi = xe^{(\alpha_l - \alpha_L)t}$ .

**From now to the end of the subsection we assume  $h(t) \equiv 0$**  for illustrative purposes; such a restriction is removed in the next subsection, which is focused on the novelties introduced by the Hardy term. We emphasize that, in any case, the result of this paper are new even for the original Laplace case, i.e. when  $h(t) \equiv 0$ . We also stress that the upper bound in the values of  $l_u$  and  $l_s$  due to  $\mathbf{gu}$ ,  $\mathbf{gs}$  disappears when  $h(t) \equiv 0$  (since  $I(0) = +\infty$ ).

**Remark 2.4.** *Assume  $h(t) \equiv 0$ ,  $\mathbf{C}$ ,  $\mathbf{gu}$ ,  $\mathbf{gs}$ , then  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$  are tangent respectively to  $y = 0$  and to  $y = -(n-2)x$  at the origin for any  $\tau \in \mathbb{R}$ .*

As in the  $t$ -independent case, all regular solutions correspond to trajectories in  $W_{l_s}^u(\tau)$ , while fast decay solutions correspond to trajectories in  $W_{l_u}^s(\tau)$ . More precisely we have the following, see [23, 24].

**Lemma 2.5.** *Assume  $h \equiv 0$ ,  $\mathbf{C}$ ,  $\mathbf{gu}$  and  $\mathbf{gs}$ . Consider the trajectory  $\mathbf{x}_{l_u}(t, \tau, \mathbf{Q})$  of  $(S_0)$  with  $l = l_u > 2_*$ , and the corresponding trajectory  $\mathbf{x}_{l_s}(t, \tau, \mathbf{R})$  of  $(S_0)$  with  $l = l_s > 2_*$ . Let  $u(r)$  be the corresponding solution of  $(Lr)$ . Then  $\mathbf{R} =$*

$$Q \exp[-(\alpha_{l_u} - \alpha_{l_s})\tau],$$

$$\begin{aligned} u(r) \text{ is a regular solution} &\iff Q \in W_{l_u}^u(\tau) \iff R \in W_{l_s}^u(\tau), \\ u(r) \text{ is a fast decay solution} &\iff Q \in W_{l_u}^s(\tau) \iff R \in W_{l_s}^s(\tau). \end{aligned}$$

We postpone the proof of the lemma to Appendix A.1, see page 36.

Note that the manifold  $W_{l_u}^u(\tau)$  is split by the origin into two connected components, one which leaves the origin and enters the  $x > 0$  semi-plane (corresponding to regular solutions  $u(r)$  positive for  $r$  small), denoted by  $W_{l_u}^{u,+}(\tau)$ , and the other which enters the  $x < 0$  semi-plane (corresponding to regular solutions  $u(r)$  negative for  $r$  small), denoted by  $W_{l_u}^{u,-}(\tau)$ . Similarly  $W_{l_s}^s(\tau)$  is split by the origin into  $W_{l_s}^{s,+}(\tau)$  and  $W_{l_s}^{s,-}(\tau)$ , which leave the origin and enter respectively in  $x > 0$  and in  $x < 0$  (and correspond to fast decay solutions  $u(r)$  which are definitively positive and definitively negative respectively).

Now we turn to consider briefly singular and slow decay solutions, see e.g. [13].

**Remark 2.6.** Assume  $h(t) \equiv 0$  and  $\mathbf{Gu}$  with  $K > 0$ , then (2.4) admits a critical point  $(P_x, P_y, 0)$  such that  $P_x > 0$ . This point admits an unstable manifold which is 1-dimensional if  $l_u \geq 2^*$  and 3-dimensional if  $2_* < l_u < 2^*$ . The trajectories  $(\mathbf{x}_{l_u}(t), z(t))$  contained in this manifold correspond to singular solutions  $v(r)$  of (Lr) such that  $\lim_{r \rightarrow 0} v(r)r^{\alpha_{l_u}} = P_x > 0$ . It is easy to check that if  $l_u > 2^*$  we have a unique singular solution, while if  $2_* < l_u < 2^*$  we have uncountably many singular solutions. Further, if  $l_u \neq 2^*$ , any trajectory  $\mathbf{x}_{l_u}(t)$  of (S) bounded for  $t \leq 0$  converges either to  $\mathbf{P} = (P_x, P_y)$  or to  $-\mathbf{P}$  or to the origin as  $t \rightarrow -\infty$ .

Assume  $h(t) \equiv 0$  and  $\mathbf{Gs}$  with  $K > 0$ , then (2.5) admits a critical point  $(P_x, P_y, 0)$  such that  $P_x > 0$ . This point admits a stable manifold which is 1-dimensional if  $2_* < l_s \leq 2^*$  and 3-dimensional if  $l_s > 2^*$ . The trajectories  $(\mathbf{x}_{l_s}(t), \zeta(t))$  contained in this manifold correspond to slow decay solutions  $v(r)$  of (Lr) such that  $\lim_{r \rightarrow \infty} v(r)r^{\alpha_{l_s}} = P_x > 0$ . If  $2_* < l_s < 2^*$  we have a unique slow decay solution, while if  $l_s > 2^*$  we have uncountably many slow decay solutions. Further, if  $l_s \neq 2^*$ , any trajectory  $\mathbf{x}_{l_s}(t)$  of (S) bounded for  $t \geq 0$  converges either to  $\mathbf{P} = (P_x, P_y)$  or to  $-\mathbf{P}$  or to the origin as  $t \rightarrow +\infty$ .

The proof follows from elementary arguments on the phase portrait, see e.g. [13, Lemma 2.9] for details.

### 2.3 Stable and unstable manifolds with Hardy potentials.

We go back to consider (Hr), and (S) where  $h(t) \not\equiv 0$  satisfies **H**. We list some results which explain similarities and differences with respect to Section 2.2. Their proofs rely on standard facts in invariant manifold theory for non-autonomous systems, and in particular on exponential dichotomy: they are postponed to the Appendix.

**Remark 2.7.** Assume  $\mathbf{gu}$ ; if  $\eta \neq 0$  regular solutions for (Hr) do not exist, due to the singularity of the equation for  $r = 0$ . They are replaced by solutions which

(may) exhibit a singular behavior as  $r \rightarrow 0$ . More precisely, for any  $d \in \mathbb{R}$  there is a unique solution,  $u(r) = u(r, d)$  of (Hr) such that  $u(r)r^{\kappa(\eta)} \rightarrow d$  as  $r \rightarrow 0$ . Analogously assume **gs**; then the behavior of fast decay solutions changes slightly. I.e., for any  $L \in \mathbb{R}$  there is a unique solution  $v(r, L)$  such that  $v(r, L)r^{n-2-\kappa(\beta)} \rightarrow L$  as  $r \rightarrow +\infty$  (cf. Definition 1.1).

In particular, Lemma 2.5 continues to hold respectively for  $\mathcal{R}$ -solutions and fd-solutions.

**Remark 2.8.** Assume **Gu** and **Gs** with  $K > 0$  and **H** (allow  $h(t) \not\equiv 0$ ). Then Remark 2.6, continues to hold almost with no differences: In particular  $\mathcal{S}$ -solutions are asymptotic to  $\pm P_x r^{\alpha_{l_u}}$  as  $r \rightarrow 0$ , while sd-solutions are asymptotic to  $\pm P_x r^{\alpha_{l_s}}$  as  $r \rightarrow +\infty$ . The only change is in the value of  $P_x$  (which however can be computed explicitly).

We also observe that if **Gu**, **Gs**, **H** hold but  $K < 0$ , then there are no  $\mathcal{S}$ -solutions neither sd-solutions, so solutions of (Hr) which are defined in a neighbourhood of  $r = 0$  and are definitively positive for  $r$  small are  $\mathcal{R}$ -solutions and the ones which are defined in a neighbourhood of  $r = \infty$  and are definitively positive for  $r$  large are fd-solutions.

Denote by  $\mathcal{A}_l(t) = \begin{pmatrix} \alpha_l & 1 \\ -h(e^t) & \gamma_l \end{pmatrix}$ . We recall that if  $\mathcal{A}_l(t) \equiv \mathcal{A}_l$  is a constant matrix (e.g. when  $h \equiv 0$  as for (Lr)), then the tangent spaces to  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$ , say  $\ell^u(\tau)$  and  $\ell^s(\tau)$ , are independent from  $\tau$ . This is not the case if  $\mathcal{A}_l(t) \not\equiv \mathcal{A}_l$ . Let  $m^u(\tau)$  and  $m^s(\tau)$  be such that

$$\begin{aligned} \ell^u(\tau) &:= \{(1, m^u(\tau)) \mid x \in \mathbb{R}\}, & \ell^u(-\infty) &:= \{(1, m^u(-\infty))x \mid x \in \mathbb{R}\}, \\ \ell^s(\tau) &:= \{(1, m^s(\tau))x \mid x \in \mathbb{R}\}, & \ell^s(+\infty) &:= \{(1, m^s(+\infty))x \mid x \in \mathbb{R}\}. \end{aligned} \quad (2.8)$$

**Remark 2.9.** Assume **gu**, **gs** and allow  $h(t) \not\equiv 0$ ; then  $\ell^u(\tau)$  and  $\ell^s(\tau)$ , change smoothly with  $\tau$ . Moreover  $m^u(\tau) \rightarrow m^u(-\infty) := -(\kappa(\eta))$  as  $\tau \rightarrow -\infty$  and  $m^s(\tau) \rightarrow m^s(+\infty) := -(n-2-\kappa(\beta))$  as  $\tau \rightarrow +\infty$ . Furthermore

$$\kappa(\eta) < \frac{n-2}{2} < n-2-\kappa(\beta).$$

**Remark 2.10.** If  $0 < \eta < \frac{(n-2)^2}{4}$ , then  $\kappa(\eta) > 0$ , hence the  $\mathcal{R}$ -solution  $u(r, d)$ , with  $d > 0$ , is in fact singular, i.e.  $\lim_{r \rightarrow 0} u(r) = +\infty$ , and accordingly  $u'(r)$  is negative and  $\lim_{r \rightarrow 0} u'(r) = -\infty$  as  $r \rightarrow 0$ . However if  $\eta < 0$  then  $\kappa(\eta) < 0$ , i.e. the  $\mathcal{R}$ -solution  $u(r, d)$ , with  $d > 0$ , is such that  $u(r, d) \rightarrow 0$  like a power as  $r \rightarrow 0$ . Moreover it is monotone increasing for  $d > 0$ , since  $\ell^u(\tau)$  lies in  $xy > 0$  for  $\tau \ll 0$ , and consequently the first branch of  $W_{l_u}^u(\tau)$  lies in  $xy > 0$  for  $\tau \ll 0$ .

## 2.4 The lack of continuability

If **C** is removed the situation becomes more complicated.

**Remark 2.11.** In this paper we are interested in functions  $f(u, r)$  which are negative for  $u$  large and either  $r$  small or  $r$  large. In these cases equation (L) may admit solutions which are not globally defined, i.e. **C** is not fulfilled. So we adopt the following notation: we say that a solution  $u(r, d)$ , resp. a solution  $v(r, L)$ , is defined in a certain maximal interval  $[0, \varrho_d)$  with  $\varrho_d \in (0, +\infty]$ , resp. in  $(\bar{\varrho}_L, +\infty)$  with  $\bar{\varrho}_L \in [0, +\infty)$ . We emphasize that  $\varrho_d \rightarrow +\infty$  as  $d \rightarrow 0$ , and  $\bar{\varrho}_L \rightarrow 0$  as  $L \rightarrow 0$ , since the null solution is continuable for any  $r \geq 0$ .

We introduce the following definitions

$$\begin{aligned} d_\tau^+ &= \sup\{D \mid \rho_d < e^\tau \text{ for any } 0 < d < D\}, \\ d_\tau^- &= \inf\{D \mid \rho_d < e^\tau \text{ for any } D < d < 0\}, \\ L_\tau^+ &= \sup\{\bar{L} \mid \bar{\rho}_L > e^\tau \text{ for any } 0 < L < \bar{L}\}, \\ L_\tau^- &= \inf\{\bar{L} \mid \bar{\rho}_L > e^\tau \text{ for any } \bar{L} < L < 0\}, \end{aligned} \quad (2.9)$$

Obviously the intervals  $(d_\tau^-, d_\tau^+)$  and  $(L_\tau^-, L_\tau^+)$  coincide with the whole  $\mathbb{R}$  if **C** is assumed, but they are bounded in the cases considered in this paper.

The lack of continuability is a relevant problem in order to apply dynamical system techniques and invariant manifold theory for non-autonomous systems. Let us denote by  $\tilde{W}_{l_u}^u(\tau)$  and  $\tilde{W}_{l_s}^s(\tau)$  the sets characterized as in (2.6). In the Appendix we will show that  $\tilde{W}_{l_u}^u(\tau)$  and  $\tilde{W}_{l_s}^s(\tau)$  may be disconnected. As usual, we can split these sets in their components  $\tilde{W}_{l_u}^{u,\pm}(\tau)$  and  $\tilde{W}_{l_s}^{s,\pm}(\tau)$ , which may be disconnected too.

For any fixed  $\tau$ , let us denote by  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$  respectively the connected component of  $\tilde{W}_{l_u}^u(\tau)$  and  $\tilde{W}_{l_s}^s(\tau)$  containing the origin, which are 1-dimensional manifolds, as we will see just below. We stress that there is no abuse of notation since, if **C** is assumed,  $\tilde{W}_{l_u}^u(\tau)$  and  $\tilde{W}_{l_s}^s(\tau)$  are 1-dimensional connected manifolds so they coincide with  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$  respectively. Similarly, we can introduce the connected branches departing from the origin  $W_{l_u}^{u,\pm}(\tau) \subset \tilde{W}_{l_u}^{u,\pm}(\tau)$  and  $W_{l_s}^{s,\pm}(\tau) \subset \tilde{W}_{l_s}^{s,\pm}(\tau)$  without abuse of notation.

**Lemma 2.12.** Assume **gu** and **gs**, then there are 1-dimensional immersed manifolds  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$  with the following properties: they contain the origin, they are connected, and they are subsets of the sets  $\tilde{W}_{l_u}^u(\tau)$  and  $\tilde{W}_{l_s}^s(\tau)$  characterized as in (2.6).

Lemma 2.5 continues to hold with the following changes.

**Lemma 2.13.** Assume **gu** and **gs**. Consider the trajectory  $\mathbf{x}_{l_u}(t, \tau, \mathbf{Q})$  of (S) with  $l = l_u \in (2_*(\eta), I(\eta))$ , and the corresponding trajectory  $\mathbf{x}_{l_s}(t, \tau, \mathbf{R})$  of (S) with  $l = l_s \in (2_*(\beta), I(\beta))$ . Let  $u(r)$  be the corresponding solution of (Hr). Then  $\mathbf{R} = \mathbf{Q} \exp[-(\alpha_{l_u} - \alpha_{l_s})\tau]$ .

$$\begin{aligned} u(r) \text{ is a } \mathcal{R}\text{-solution} &\iff \mathbf{Q} \in \tilde{W}_{l_u}^u(\tau) \iff \mathbf{R} \in \tilde{W}_{l_s}^u(\tau), \\ u(r) \text{ is a fd-solution} &\iff \mathbf{Q} \in \tilde{W}_{l_u}^s(\tau) \iff \mathbf{R} \in \tilde{W}_{l_s}^s(\tau). \end{aligned}$$

Recall that  $W_{l_u}^u(\tau) \subset \tilde{W}_{l_u}^u(\tau)$ ,  $W_{l_s}^u(\tau) \subset \tilde{W}_{l_s}^u(\tau)$ . Consider a  $\mathcal{R}$ -solution  $u(r)$ , then we can find  $N \gg 0$  such that  $\mathbf{Q} \in W_{l_u}^u(\tau)$  and  $\mathbf{R} \in W_{l_s}^u(\tau)$  for any

$\tau < -N$ . Similarly consider a fd-solution  $u(r)$ , then we can find  $N \gg 0$  such that  $\mathbf{Q} \in W_{l_u}^s(\tau)$  and  $\mathbf{R} \in W_{l_s}^s(\tau)$  for any  $\tau > N$ .

The proofs of the previous lemmas are postponed to Appendix A.1.

**Remark 2.14.** We stress that the tangents to  $\tilde{W}_{l_u}^u(\tau)$  and to  $\tilde{W}_{l_s}^s(\tau)$  in the origin coincide by construction with the tangents to  $W_{l_u}^u(\tau)$  and to  $W_{l_s}^s(\tau)$ . Hence, in Remark 2.4 we can remove assumption **C**.

## 2.5 Kelvin inversion and Fowler transformation

An important tool in the investigation of equations like (Hr) is a change of variables known as Kelvin inversion. Let us set

$$s = 1/r, \quad \tilde{u}(s) = u(1/s)s^{2-n}, \quad \tilde{f}(\tilde{u}, s) = f(\tilde{u}s^{n-2}, 1/s)s^{-2-n}. \quad (2.10)$$

From a straightforward computation we see that  $u(r)$  satisfies (Hr) if and only if  $\tilde{u}(s)$  satisfies the following equation:

$$\frac{d}{ds}[\tilde{u}_s(s)s^{n-1}] + \tilde{f}(\tilde{u}, s)s^{n-1} = 0. \quad (2.11)$$

In particular, if  $f$  is of type (1.1) then  $\tilde{f}(u, s) = k(1/s)s^{(n-2)(q-2^*)}u|u|^{q-2}$ .

We stress that  $\mathcal{R}$ -solutions  $u(r, d)$  of (Hr) are driven by (2.10) into fd-solutions  $\tilde{v}(s, d) = u(1/s, d)s^{2-n}$  of (2.11), while fd-solutions  $v(r, L)$  of (Hr) are driven into  $\mathcal{R}$ -solutions  $\tilde{u}(s, L) = v(1/s, L)s^{2-n}$  of (2.11); we emphasize that  $d = \lim_{r \rightarrow 0} u(r)r^{\kappa(\eta)} = \lim_{s \rightarrow +\infty} \tilde{v}(s)s^{n-2-\kappa(\eta)}$ , and  $L = \lim_{r \rightarrow +\infty} v(r)r^{n-2-\kappa(\beta)} = \lim_{s \rightarrow 0} \tilde{u}(s)s^{n-2-\kappa(\beta)}$ . It is important to observe that generically if  $f$  is subcritical (respectively supercritical) then  $\tilde{f}$  is supercritical (resp. subcritical), see, e.g., [23, §2] for the analogous statement for (Lr).

We emphasize that Kelvin inversion (2.10) assumes a more clear form when it is combined with Fowler transformation (2.1). In fact, when we apply (2.1) to (2.11), by setting

$$\begin{cases} \tau = -t \\ \tilde{x}_l(\tau) = \tilde{u}(e^\tau)e^{-\gamma_l\tau} = u(e^{-\tau})e^{-\alpha_l\tau} = u(e^t)e^{\alpha_l t} \\ \tilde{y}_l(\tau) = \tilde{u}'(e^\tau)e^{(-\gamma_l+1)\tau} = -u'(e^t)e^{(\alpha_l+1)t} - (n-2)u(e^t)e^{\alpha_l t} \end{cases} \quad (2.12)$$

we simply pass from system (S) to the following one:

$$\begin{pmatrix} \frac{\partial \tilde{x}_l}{\partial \tau} \\ \frac{\partial \tilde{y}_l}{\partial \tau} \end{pmatrix} = \begin{pmatrix} -\gamma_l & 1 \\ -h(e^{-\tau}) & -\alpha_l \end{pmatrix} \begin{pmatrix} \tilde{x}_l \\ \tilde{y}_l \end{pmatrix} + \begin{pmatrix} 0 \\ -g_l(\tilde{x}_l, -\tau) \end{pmatrix}. \quad (2.13)$$

We stress that (2.13) is obtained from (S) simply by changing the values of the parameters  $(\alpha_l, \gamma_l)$  into  $(-\gamma_l, -\alpha_l)$ , and evaluating the functions  $g_l(x, t)$  and  $h(e^t)$  in  $-\tau$  in spite of  $\tau$ . We give the details of the computation for reader's



convenience. Let us set  $f_h(u, r) := \frac{h(r)}{r^2}u + f(u, r)$  and introduce  $\tilde{f}_h$  as in (2.10), then

$$\begin{aligned}\frac{\partial}{\partial \tau} \tilde{x}_l(\tau) &= -\gamma_l \tilde{u}(e^\tau) e^{-\gamma_l \tau} + \tilde{u}'(e^\tau) e^{(-\gamma_l+1)\tau} = -\gamma_l \tilde{x}_l(\tau) + \tilde{y}_l(\tau) \\ \frac{\partial}{\partial \tau} \tilde{y}_l(\tau) &= \frac{\partial}{\partial \tau} [(\tilde{y}_l(\tau) e^{\alpha_l \tau}) e^{-\alpha_l \tau}] = -\alpha_l \tilde{y}_l(\tau) + e^{-\alpha_l \tau} \frac{\partial}{\partial \tau} [\tilde{u}'(e^\tau) e^{(n-1)\tau}] \\ &= -\alpha_l \tilde{y}_l(\tau) - \tilde{f}_h(\tilde{u}(e^\tau), e^\tau) e^{(n-\alpha_l)\tau} \\ &= -\alpha_l \tilde{y}_l(\tau) - f_h(\tilde{u}(e^\tau) e^{(n-2)\tau}, e^{-\tau}) e^{-(\alpha_l+2)\tau} \\ &= -\alpha_l \tilde{y}_l(\tau) - f_h(\tilde{x}_l(\tau) e^{\alpha_l \tau}, e^{-\tau}) e^{-(\alpha_l+2)\tau} \\ &= -\alpha_l \tilde{y}_l(\tau) - h(e^{-\tau}) \tilde{x}_l(\tau) - g_l(\tilde{x}_l(\tau), -\tau).\end{aligned}$$

Let us assume that  $f(u, r)$  satisfies **Gu** with  $l_u = \bar{l}_u > 2_*$  (respectively **Gs** with  $l_s = \bar{l}_s > 2_*$ ) then  $\tilde{f}(u, r)$  satisfies **Gs** with  $l_s = \bar{L}_s > 2_*$  (resp. **Gu** with  $l_u = \bar{L}_u > 2_*$ ), where

$$\bar{L}_s = 2 - \frac{2}{\gamma_{\bar{l}_u}} = \frac{2[\bar{l}_u(n-1) - 2n]}{\bar{l}_u(n-2) - 2n + 2}; \quad \bar{L}_u = 2 - \frac{2}{\gamma_{l_s}} = \frac{2[\bar{l}_s(n-1) - 2n]}{\bar{l}_s(n-2) - 2n + 2}$$

In particular we have  $\alpha_{\bar{L}_s} = -\gamma_{\bar{l}_u}$  and  $\gamma_{\bar{L}_s} = -\alpha_{\bar{l}_u}$  (resp.  $\alpha_{\bar{L}_u} = -\gamma_{\bar{l}_s}$  and  $\gamma_{\bar{L}_u} = -\alpha_{\bar{l}_s}$ ).

We emphasize that, if  $g_l(x, T)$  is  $T$ -independent, the condition  $\alpha_l + \gamma_l > 0$  means that (Hr) is subcritical (respectively  $\alpha_l + \gamma_l < 0$  and  $\alpha_l + \gamma_l = 0$  mean (Hr) supercritical and critical). Hence, it is clear that when we pass from (S) to (2.13), the unstable manifold  $W^u(T)$  is driven into the stable manifold  $W^s(-T)$  and viceversa, and a subcritical system is driven into a supercritical system.

Moreover we emphasize that, in presence of a Hardy potential, we have e.g.

$$2_*(\beta) < \bar{l}_u < 2^* \quad \Rightarrow \quad 2^* < \bar{L}_s < I(\beta). \quad (2.14)$$

## 2.6 Statement of the results.

Let us introduce some further assumptions we will assume together with **gu** and **gs**.

**L1** There is  $\mathfrak{T} \in \mathbb{R}$ , such that  $\frac{g_{l_u}(x, t)}{x} \leq 0$  for any  $x \in \mathbb{R}$  and any  $t < \mathfrak{T}$  and  $\liminf_{|x| \rightarrow +\infty} \frac{g_{l_s}(x, t)}{x} \geq 0$  for any  $t > \mathfrak{T}$ .

**L2** There is  $\mathfrak{T} \in \mathbb{R}$ , such that  $\frac{g_{l_s}(x, t)}{x} \leq 0$  for any  $x \in \mathbb{R}$  and any  $t > \mathfrak{T}$  and  $\liminf_{|x| \rightarrow +\infty} \frac{g_{l_u}(x, t)}{x} \geq 0$  for any  $t < \mathfrak{T}$ .

We stress that **L1** is trivially satisfied if  $f$  is as in (1.2a) and **K** holds, just setting  $\mathfrak{T} = \ln R$ . By symmetry, if  $f$  is as in (1.2b) and **K** holds, **L2** follows.

We are now ready to state the main results, using the terminology introduced in Definition 1.1.

**Theorem 2.15.** Consider (Hr) and assume  $\mathbf{Gs}$  with  $K > 0$  and  $l_s \in (2^*, I(\beta))$ ,  $\mathbf{gu}$  and **L1**. Then there are four positive strictly increasing sequences  $(A_k^\pm)_{k \geq 0}$  and  $(B_k^\pm)_{k \geq 0}$  such that  $u(r, A_0^+) = v(r, B_0^+)$  is a positive  $\mathcal{R}^0 \text{fd}$ , respectively  $u(r, -A_0^-) = v(r, -B_0^-)$  is a negative  $\mathcal{R}^0 \text{fd}$ .

For any  $k > 1$ ,  $u(r, \pm A_k^\pm)$  is a  $\mathcal{R}^k \text{fd}$ . In particular we have,  $u(r, \pm A_{2j}^\pm) = v(r, \pm B_{2j}^\pm)$  and  $u(r, \pm A_{2j+1}^\pm) = v(r, \mp B_{2j+1}^\mp)$ . Moreover  $u(r, d)$  is a positive  $\mathcal{R}^0 \text{sd}$  for any  $0 < d < A_0^+$ , and for any  $k > 0$ , there is  $a_k^+ \in [A_{k-1}^+, A_k^+)$  such that  $u(r, d)$  is a  $\mathcal{R}^k \text{sd}$  whenever  $d \in (a_k^+, A_k^+)$ . An analogous statement holds for  $\mathcal{R}^- \text{sd}$   $u(r, d)$  where  $d < 0$ .

We recall that if  $f$  satisfies  $\mathbf{Gs}$  with  $K > 0$  and  $l_s \in (2^*, I(\beta))$ ,  $\mathbf{gu}$  and **L1**, then  $\tilde{f}$  obtained via (2.10) satisfies  $\mathbf{Gu}$  with  $K > 0$  and  $l_u \in (2_*(\beta), 2^*)$ ,  $\mathbf{gs}$  and **L2**. Moreover  $\mathcal{R}$ -solutions are turned into  $\text{fd}$ -solutions and viceversa, and  $\mathcal{S}$ -solutions into  $\text{sd}$ -solutions, see Subsection 2.4. So, applying Kelvin inversion on Theorem 2.15, we obtain the following result.

**Theorem 2.16.** Consider (Hr) and assume  $\mathbf{Gu}$  with  $K > 0$  and  $l_u \in (2_*(\eta), 2^*)$ ,  $\mathbf{gs}$  and **L2**. Then there are four positive strictly increasing sequences  $(A_k^\pm)_{k \geq 0}$  and  $(B_k^\pm)_{k \geq 0}$  such that  $u(r, A_0^+) = v(r, B_0^+)$  is a positive  $\mathcal{R}^0 \text{fd}$ , while  $u(r, -A_0^-) = v(r, -B_0^-)$  is a negative  $\mathcal{R}^0 \text{fd}$ .

For any  $k > 1$ ,  $u(r, \pm A_k^\pm)$  is a  $\mathcal{R}^k \text{fd}$ . In particular we have,  $u(r, \pm A_{2j}^\pm) = v(r, \pm B_{2j}^\pm)$  and  $u(r, \pm A_{2j+1}^\pm) = v(r, \mp B_{2j+1}^\mp)$ .

Moreover  $v(r, L)$  is a positive  $\mathcal{S}^0 \text{fd}$  for any  $0 < L < B_0^+$ , and for any  $k > 0$ , there is  $b_k^+ \in [B_{k-1}^+, B_k^+)$  such that  $v(r, L)$  is a  $\mathcal{S}^k \text{fd}$  whenever  $L \in (b_k^+, B_k^+)$ . An analogous statement holds for  $\mathcal{S}^- \text{fd}$   $v(r, L)$  where  $L < 0$ .

### 3 Proofs.

#### 3.1 Preliminary lemmas.

For every solution  $\mathbf{x}_l(t) = (x_l(t), y_l(t))$  of (S), we introduce polar coordinates

$$\rho_l = \|\mathbf{x}_l\|, \quad \phi_l = \arctan(y_l/x_l). \quad (3.1)$$

Taking into account (2.1), we stress that if we switch between different values of  $l$ , say  $l_1$  and  $l_2$ , we get  $\rho_{l_2} = \exp[(\alpha_{l_2} - \alpha_{l_1})t]\rho_{l_1}$  and  $\phi_{l_1}(t) = \phi_{l_2}(t)$ , so we can drop the subscript in  $\phi$ .

In particular, the next remark easily follows from the fact that the flow on the  $y$ -axis rotates clockwise, i.e.,

$$\dot{x}_l(t)y_l(t) > 0 \quad \text{when } x_l(t) = 0 \text{ and } y_l(t) \neq 0. \quad (3.2)$$

Let us denote by  $\text{Int}[x]$  the integer part of  $x$ .

**Remark 3.1.** Consider the trajectory of a solution  $\mathbf{x}_l(t)$  of (S); then  $\text{Int}[\frac{1}{2} + \phi(t)/\pi]$  is decreasing in  $t$ .

Let us denote by  $\Theta^u(\tau) = \arctan(m^u(\tau))$  and by  $\Theta^s(\tau) = \arctan(m^s(\tau))$ , see (2.8); we assume w.l.o.g. that these functions are continuous, and  $\Theta^u(\tau) \rightarrow \Theta^u(-\infty)$  and  $\Theta^s(\tau) \rightarrow \Theta^s(+\infty)$  as  $\tau \rightarrow -\infty$  and as  $\tau \rightarrow +\infty$  respectively. Then we have the following.

**Lemma 3.2.** *Assume  $\mathbf{Gs}$  with  $K > 0$  and  $l_s \in (2^*, I(\beta))$ , then  $m^s(\tau) < -\frac{n-2}{2}$  for any  $\tau \in \mathbb{R}$ .*

*Proof.* From Remark 2.9 we have  $m^s(+\infty) = -(n-2-\kappa(\beta)) < -\frac{n-2}{2}$  so that the conclusion easily follows from  $\mathbf{Gs}$  for  $\tau$  sufficiently large, say  $\tau \geq T_2$ . Moreover by  $\mathbf{Gs}$  we see that for any  $\tau \in \mathbb{R}$  we have  $g_{l_s}(x, t) \leq A(t)x\Delta(x)$  for any  $t \geq \tau$ , where  $\Delta(x)$  is a continuous increasing function such that  $\Delta(0) = 0$ , and  $A(t)$  is a continuous function such that  $\lim_{t \rightarrow +\infty} A(t)$  is positive and finite. Therefore there is  $\delta(\tau) > 0$  such that

$$\frac{g_{l_s}(x, t)}{x} < \frac{(n-2)^2}{4} - h(e^t) \quad \text{if } t \geq \tau \text{ and } |x| \leq \delta(\tau). \quad (3.3)$$

Let us now consider the triangle  $\mathcal{T}(\tau)$  having vertices  $\mathbf{O} = (0, 0)$ ,  $\mathbf{A}(\tau) = (\delta(\tau), -\frac{n-2}{2}\delta(\tau))$ ,  $\mathbf{B}(\tau) = (0, -\frac{n-2}{2}\delta(\tau))$ , and denote by  $o(\tau), a(\tau), b(\tau)$  the edges opposite to  $\mathbf{O}, \mathbf{A}(\tau), \mathbf{B}(\tau)$ , without endpoints.

If  $\mathbf{x}_{l_s}(t_o) \in b(\tau)$ , for  $t_o \geq \tau$ , applying (3.3) we find

$$\begin{aligned} & \left. \frac{d}{dt} \left( y_{l_s} + \frac{n-2}{2} x_{l_s} \right) \right|_{t=t_o} \\ &= \frac{n-2}{2} x_{l_s}(t_o) \left( \alpha_{l_s} - \gamma_{l_s} - \frac{n-2}{2} \right) - h(e^{t_o}) x_{l_s}(t_o) - g_{l_s}(x_{l_s}(t_o), t_o) \\ &= x_{l_s}(t_o) \left[ \frac{(n-2)^2}{4} - h(e^{t_o}) - \frac{g_{l_s}(x_{l_s}(t_o), t_o)}{x_{l_s}(t_o)} \right] > 0. \end{aligned} \quad (3.4)$$

Thus the flow on  $b(\tau)$  points towards the exterior of  $\mathcal{T}(\tau)$ , whenever  $t \geq \tau$ . Moreover by construction the flow of (S) on  $a(\tau)$  points towards the exterior of  $\mathcal{T}(\tau)$ . Finally observe that if  $\mathbf{x}_{l_s}(t_o) \in o(\tau)$  for  $t_o \geq \tau$ , we have

$$\left. \frac{d}{dt} y_{l_s} \right|_{t=t_o} \geq \left[ \frac{n-2}{2} |\gamma_{l_s}| - h(e^{t_o}) - \frac{g_{l_s}(x_{l_s}(t_o), t_o)}{x_{l_s}(t_o)} \right] x_{l_s}(t_o) > 0 \quad (3.5)$$

where we have used  $|\gamma_{l_s}| > \frac{n-2}{2}$ , being  $l_s > 2^*$ .

So we can apply Lemmas A.5 and A.6, thus finding that there is a connected subset  $\bar{W}^s(\tau) \subset T(\tau) \cap W_{l_s}^s(\tau)$  containing the origin and a point in  $o(\tau)$ . It follows that locally  $\ell^s(\tau) \subset T(\tau)$  too, therefore  $m^s(\tau) < -\frac{n-2}{2}$ .  $\square$

Assume that we are in the hypotheses of Theorem 2.15.

From  $\mathbf{Gs}$  the manifolds  $W_{l_s}^s(\tau)$  exist for any  $\tau \in (-\infty, +\infty]$ . Moreover from  $\mathbf{gu}$  we see that  $W_{l_u}^u(\tau)$  exists for any  $t \in \mathbb{R}$  and its tangent at the origin is  $y = -m^u(\tau)x$  where  $m^u(\tau) \rightarrow m^u(-\infty) = \arctan(-\kappa(\eta))$  as  $\tau \rightarrow -\infty$ , see Remark 2.9.

Let  $\Sigma_{l_u}^{u,\pm}(\cdot, \tau) : [0, +\infty) \rightarrow W_{l_u}^{u,\pm}(\tau)$  and  $\Sigma_{l_s}^{s,\pm}(\cdot, \tau) : [0, +\infty) \rightarrow W_{l_s}^{s,\pm}(\tau)$  be smooth parameterizations respectively of  $W_{l_u}^{u,\pm}(\tau)$  and  $W_{l_s}^{s,\pm}(\tau)$  such that  $\Sigma_{l_u}^{u,\pm}(0, \tau) = (0, 0)$  and  $\Sigma_{l_s}^{s,\pm}(0, \tau) = (0, 0)$ . We assume w.l.o.g. that the functions  $\Sigma_{l_u}^{u,\pm} : [0, +\infty) \times (-\infty, +\infty) \rightarrow \mathbb{R}^2$  and  $\Sigma_{l_s}^{s,\pm} : [0, +\infty) \times (-\infty, +\infty) \rightarrow \mathbb{R}^2$  are continuous in both the variables.

The following result establishes a correspondence between trajectories of (S) and solutions of (Hr).

**Lemma 3.3.** *Assume the hypotheses of Theorem 2.15 and fix  $T \in \mathbb{R}$ . Let  $u(r, d(\omega))$  be the  $\mathcal{R}$ -solution of (Hr) corresponding to  $\mathbf{x}_{l_u}(t, T, \Sigma_{l_u}^{u,+}(\omega, T))$ . Then,  $d(\omega) \geq 0$  is a strictly increasing function such that  $d(0) = 0$ . Moreover, let  $v(r, L(\sigma))$  be the fd-solution of (Hr) corresponding to  $\mathbf{x}_{l_s}(t, T, \Sigma_{l_s}^{s,+}(\sigma, T))$ . Then,  $L(\sigma)$  is a strictly increasing function such that  $L(0) = 0$ .*

The proof is postponed to Appendix A.2, however see [13, Lemma 2.10] for the simpler case where  $h(r) \equiv 0$  and  $\mathbf{C}$  is assumed.

Lemma 3.3 permits us to introduce an additional parametrization on our manifolds depending on the parameters  $d$  and  $L$ .

**Remark 3.4.** *For every  $\tau \in \mathbb{R}$ , we can parametrize  $W_{l_u}^{u,+}(\tau)$  directly with  $d$  and  $W_{l_s}^{s,+}(\tau)$  with  $L$ . In this way we can introduce the new parametrizations  $\Upsilon_{l_u}^{u,+}(\cdot, \tau) : [0, d_\tau^+) \rightarrow W_{l_u}^{u,+}(\tau)$  and  $\Upsilon_{l_s}^{s,+}(\cdot, \tau) : [0, L_\tau^+) \rightarrow W_{l_s}^{s,+}(\tau)$ , which are continuous in both the variables; here  $d_\tau^+, L_\tau^+ \in (0, +\infty]$  have been defined in (2.9). However in this case  $\Upsilon_{l_u}^{u,+}$  cannot be extended continuously to  $\tau = -\infty$  and  $\Upsilon_{l_s}^{s,+}$  cannot be extended to  $\tau = +\infty$ , since  $\Upsilon_{l_u}^{u,+}(d, \tau) \rightarrow (0, 0)$  as  $\tau \rightarrow -\infty$  and  $\Upsilon_{l_s}^{s,+}(L, \tau) \rightarrow (0, 0)$  as  $\tau \rightarrow +\infty$ , for any  $d \in (d_\tau^-, d_\tau^+)$  and  $L \in (L_\tau^-, L_\tau^+)$ .*

*We underline that, in general, we do not have  $d_\tau^+ = +\infty$  or  $L_\tau^+ = +\infty$ . E.g., by Remark 2.11, once fixed  $\tau \in \mathbb{R}$ , if  $\varrho_d < e^\tau$  for a certain  $d > 0$  then  $d_\tau^+ \leq d$ . An analogous statement holds for  $W_{l_u}^{u,-}(\tau)$  and  $W_{l_s}^{s,-}(\tau)$ .*

**Remark 3.5.** *Assumption L1 guarantees both forward and backward continuity of the trajectories of (S) for  $t \geq \mathfrak{T}$ . Therefore  $d_\tau^+ = d_\tau^\pm$ , and  $L_\tau^\pm = \pm\infty$  for any  $\tau \geq \mathfrak{T}$ .*

Using (2.7) we can define for  $W_{l_u}^{s,\pm}(\tau)$  the continuous parameterizations  $\Sigma_{l_u}^{s,\pm}(\omega, \tau) = \Sigma_{l_s}^{s,\pm}(\omega, \tau)e^{(\alpha_{l_u} - \alpha_{l_s})\tau}$ . It is straightforward to check that the property explained in Lemma 3.3 holds for  $\Sigma_{l_u}^{s,\pm}(\omega, \tau)$ , too.

We need to consider also the following parametrizations in polar coordinates of the manifolds:

$$\begin{aligned} \Sigma_l^{u,\pm}(\omega, \tau) &= \rho^{u,\pm}(\omega, \tau; l) \left( \cos(\theta^{u,\pm}(\omega, \tau)), \sin(\theta^{u,\pm}(\omega, \tau)) \right), \\ \Sigma_l^{s,\pm}(\sigma, \tau) &= \rho^{s,\pm}(\sigma, \tau; l) \left( \cos(\theta^{s,\pm}(\sigma, \tau)), \sin(\theta^{s,\pm}(\sigma, \tau)) \right); \end{aligned} \quad (3.6)$$

$$\begin{aligned} \Upsilon_l^{u,\pm}(d, \tau) &= R^{u,\pm}(d, \tau; l) \left( \cos(\Theta^{u,\pm}(d, \tau)), \sin(\Theta^{u,\pm}(d, \tau)) \right), \\ \Upsilon_l^{s,\pm}(L, \tau) &= R^{s,\pm}(L, \tau; l) \left( \cos(\Theta^{s,\pm}(L, \tau)), \sin(\Theta^{s,\pm}(L, \tau)) \right). \end{aligned} \quad (3.7)$$

We recall that the angular coordinate of the parametrizations does not depend on the choice of  $l$  as stated in Lemma 2.5.

Let  $\mathbf{Q} \in W_l^{s,+}(\tau)$  and consider the trajectory  $\mathbf{x}_l(t, \tau, \mathbf{Q})$  of (S). Note that  $\frac{\mathbf{x}_l(t, \tau, \mathbf{Q})}{|\mathbf{x}_l(t, \tau, \mathbf{Q})|}$  approaches  $\ell^s(+\infty)$  as  $t \rightarrow +\infty$ , see (2.8). Using the polar coordinates introduced in (3.1), let us set

$$\mathbf{x}_l(t, \tau, \mathbf{Q}) = \rho_l(t, \tau, \mathbf{Q}) (\cos(\phi(t, \tau, \mathbf{Q})), \sin(\phi(t, \tau, \mathbf{Q}))), \quad (3.8)$$

where we can assume that the angular coordinate  $\phi$  satisfies  $\lim_{t \rightarrow +\infty} \phi(t, \tau, \mathbf{Q}) = -\arctan(n-2-\kappa(\beta))$ . Similarly, if we consider  $\mathbf{R} \in W_l^{u,+}(\tau)$  with the trajectory  $\mathbf{x}_l(t, \tau, \mathbf{R})$ , then  $\frac{\mathbf{x}_l(t, \tau, \mathbf{R})}{|\mathbf{x}_l(t, \tau, \mathbf{R})|}$  approaches  $\ell^u(-\infty)$  as  $t \rightarrow -\infty$ , and we can assume  $\lim_{t \rightarrow -\infty} \phi(t, \tau, \mathbf{R}) = -\arctan(\kappa(\eta))$ . However, if  $\mathbf{Q} \in W_l^{u,+}(\tau) \cap W_l^{s,+}(\tau)$  for a certain  $l$ , we must choose one of the two conditions, the other will be satisfied up to a multiple of  $2\pi$ .

### 3.2 Proof of the main theorems.

In this section we provide the proof of Theorem 2.15. The proof is based on some geometrical observations on the phase portrait. Then Theorem 2.16 follows from Kelvin inversion.

We recall that the manifolds  $W_{l_s}^{s,+}(\tau)$  are sets of initial conditions converging to the origin and a priori they are not graphs of solutions unless the system is autonomous. However, for system (S) the number of rotations around the origin performed by  $W_{l_s}^{s,+}(\tau)$  from the origin until a point  $\mathbf{Q} \in W_{l_s}^{s,+}(\tau)$ , equals the number of rotations performed by the trajectory  $\mathbf{x}_{l_s}(t, \tau, \mathbf{Q})$  for  $t \geq \tau$ , with reversed sign.

More precisely we have the following property, the proof is adapted from [13, Propositions 3.5, 3.8] (we refer also to [3, 24, 30] for more details) and it is postponed to Appendix A.2.

**Lemma 3.6.** *Let us consider system (S) and assume  $\mathbf{G}s$  with  $K > 0$  and  $l_s \in (2^*, \mathbf{I}(\beta))$ . Consider the trajectory in (3.8) with  $\mathbf{Q} = \Sigma_{l_s}^{s,+}(\sigma, \tau)$ , using the notation in (3.6). Then if  $h$  is a constant, the angle  $\theta := \theta^{s,+}(\sigma, \tau) - \theta^{s,+}(0, \tau)$  performed by the stable manifold  $W_{l_s}^{s,+}(\tau)$  equals, but with reversed sign, the angle  $\phi := \phi(+\infty, \tau, \mathbf{Q}) - \phi(\tau, \tau, \mathbf{Q})$  performed by the trajectory  $\mathbf{x}_{l_s}(t, \tau, \mathbf{Q})$ . If  $h$  is a function, the difference is*

$$\begin{aligned} |\theta - (-\phi)| &= |-\theta^{s,+}(0, \tau) + \theta^{s,+}(0, +\infty)| \\ &= |\arctan(m^s(\tau)) - \arctan(n-2-\kappa(\beta))| \leq \pi. \end{aligned} \quad (3.9)$$

We wish to underline that a similar statement can be obtained for the unstable manifold too. Moreover, notice that Lemma 3.6 is independent from the parametrization of the stable manifold, so we can use the one defined in (3.7).

**Lemma 3.7.** *Assume  $\mathbf{G}s$  with  $K > 0$ ,  $l_s \in (2^*, \mathbf{I}(\beta))$  and  $\mathbf{L1}$ , then  $W_{l_s}^{s,+}(\tau)$  and  $W_{l_s}^{s,-}(\tau)$  are spirals rotating indefinitely counterclockwise around the origin for any  $\tau \geq \mathfrak{T}$ ; therefore  $\lim_{\sigma \rightarrow +\infty} \theta^{s,+}(\sigma, \tau) = +\infty$ , and  $\lim_{\sigma \rightarrow +\infty} \theta^{s,-}(\sigma, \tau) = +\infty$  for any  $\tau \geq \mathfrak{T}$ .*

*Proof.* We prove the Lemma just for  $W_{l_s}^{s,+}(\tau)$ ; the case of  $W_{l_s}^{s,-}(\tau)$  can be obtained analogously. The Lemma is known if the system is autonomous, so we have it trivially for  $\tau = +\infty$ . In fact the manifold  $W_{l_s}^{s,+}(+\infty)$  coincides with the stable manifold  $M^{s,+}$  of the autonomous system (S) where  $g_{l_s}(x, t) \equiv Kg_{l_s}^{+\infty}(x)$ . We recall that from **L1** we get continuability of the solutions for any  $t \geq \mathfrak{T}$ . From Remark 2.2, we see that for any integer  $M > 0$  there is  $T > 0$  such that  $W_{l_s}^{s,+}(\tau)$  crosses transversally the coordinate axes, and performs at least  $M$  complete rotations counterclockwise, for any  $\tau \geq T$ . In particular, using (3.7), there exists  $L_M$  such that  $\Theta^{s,+}(L_M, T) = 2\pi M + \pi/2$ . Call  $\mathbf{Q}_M = \Upsilon_{l_s}^{s,+}(L_M, T)$  which belongs to the positive  $y$ -semiaxis. In particular  $\Upsilon_{l_s}^{s,+}([0, L_M] \times \{T\})$  performs more than  $M$  complete rotations.

Let  $\phi(t, T, \mathbf{Q}_M)$  be the angular coordinate of  $\mathbf{x}_{l_s}(t, T, \mathbf{Q}_M)$ , see (3.8), then from Lemma 3.6 we see that  $\phi(T, T, \mathbf{Q}_M) = \Theta^{s,+}(L_M, T) = 2\pi M + \pi/2$ . By (3.2) and Remark 3.1, we have,  $\phi(\tau, T, \mathbf{Q}_M) \geq 2\pi M + \pi/2$  for any  $\tau \leq T$ , as long as  $\phi(\tau, T, \mathbf{Q}_M)$  exists, therefore at least for  $\tau \in [\mathfrak{T}, T]$ . Then using again Lemma 3.6 and Lemma 3.2, we see that

$$\Theta^{s,+}(L_M, \tau) - \Theta^{s,+}(0, \tau) \geq 2\pi M + \frac{\pi}{2} - \arctan(m^s(\tau)) \geq \left(2 + \frac{1}{2}\right) \pi M,$$

thus obtaining that  $\Upsilon_{l_s}^{s,+}([0, L_M] \times \{\tau\})$  draws more than  $M$  rotations for every  $\tau \geq \mathfrak{T}$ . We have thus proved the Lemma, since  $M$  is arbitrarily large.  $\square$

**Lemma 3.8.** Assume **gu** and **L1**; then  $\limsup_{\omega \rightarrow +\infty} \rho^{u,\pm}(\omega, \tau; l_u) = +\infty$ , and

$$\begin{aligned} -\arctan\left(\frac{n-2}{2}\right) &< \theta^{u,+}(\omega, \tau) < \pi/2 \\ -\pi - \arctan\left(\frac{n-2}{2}\right) &< \theta^{u,-}(\omega, \tau) < -\pi/2 \end{aligned}$$

for every  $\omega > 0$  and for any  $\tau \leq \mathfrak{T}$ .

*Proof.* As usual, we give the proof for  $W_{l_u}^{u,+}(\tau)$ , the other follows similarly. Let

$$S(\xi) = \sup \left( \left\{ -h(e^t) - \frac{g_{l_u}(x, t)}{x} : 0 < x \leq \xi, t \leq \mathfrak{T} \right\} \cup \{0\} \right).$$

Notice that  $S(\xi)$  is increasing, and by **gu**,  $S(\xi) < +\infty$  for any fixed  $\xi$ . Let  $\mathbf{m}(\xi) = \sqrt{|S(\xi)|}$ , and set  $\mathbf{A}(\xi) = (\xi, \mathbf{m}(\xi)\xi)$ ,  $\mathbf{B}(\xi) = (\xi, -\frac{n-2}{2}\xi)$ .

This proof is analogous to the one of Lemma 3.2 and relies on Lemma A.6. We construct the triangle  $Z(\xi)$  with vertices  $\mathbf{O}$ ,  $\mathbf{A}(\xi)$ ,  $\mathbf{B}(\xi)$ , and with edges  $o(\xi)$ ,  $a(\xi)$ ,  $b(\xi)$  opposite to the vertices  $\mathbf{O}$ ,  $\mathbf{A}(\xi)$ ,  $\mathbf{B}(\xi)$  respectively.

Let  $\mathbf{x}_{l_u}(t_o) \in b(\xi)$  at a certain time  $t_o \leq \mathfrak{T}$ , then

$$\begin{aligned} &\left. \frac{d}{dt} (y_{l_u} - \mathbf{m}(\xi)x_{l_u}) \right|_{t=t_o} \\ &= - \left( \mathbf{m}(\xi)(\mathbf{m}(\xi) + n - 2) + h(e^{t_o}) + \frac{g_{l_u}(x_{l_u}(t_o), t_o)}{x_{l_u}(t_o)} \right) x_{l_u}(t_o) \\ &< - \left( S(\xi) + h(e^{t_o}) + \frac{g_{l_u}(x_{l_u}(t_o), t_o)}{x_{l_u}(t_o)} \right) x_{l_u}(t_o) \leq 0. \end{aligned} \quad (3.10)$$

From **L1** we see that if  $\mathbf{x}_{l_u}(t_o) \in a(\xi)$  at a certain time  $t_o \leq \mathfrak{T}$ , then

$$\begin{aligned} & \left. \frac{d}{dt} \left( y_{l_u} + \frac{n-2}{2} x_{l_u} \right) \right|_{t=t_o} \\ &= \frac{n-2}{2} x_{l_u}(t_o) \left( \alpha_{l_u} - \gamma_{l_u} - \frac{n-2}{2} \right) - h(e^{t_o}) x_{l_u}(t_o) - g_{l_u}(x_{l_u}(t_o), t_o) \quad (3.11) \\ &= x_{l_u}(t_o) \left[ \frac{(n-2)^2}{4} - h(e^{t_o}) - \frac{g_{l_u}(x_{l_u}(t_o), t_o)}{x_{l_u}(t_o)} \right] > 0. \end{aligned}$$

So from (3.10), (3.11) the flow of (S) on  $a(\xi) \cup b(\xi)$ , points towards the interior of  $Z(\xi)$ , for any  $\tau \leq \mathfrak{T}$ .

Assume first  $l_u > 2^*$  so that on  $o(\xi)$  we have  $\dot{x}_{l_u} > 0$ . The flow on  $o(\xi)$  points towards the exterior of  $Z(\xi)$ ; hence we can apply Lemma A.6, and we find a connected subset  $\mathcal{W}(\tau) \subset (W_{l_u}^u(\tau) \cap Z(\xi))$  containing  $\mathbf{O}$  and a point in  $o(\xi)$ , for any  $\xi > 0$  and any  $\tau \leq \mathfrak{T}$ . Such a procedure can be repeated for  $\xi$  arbitrarily large, thus concluding the proof of the lemma.

Now assume  $l_u \in (2_*(\eta), 2^*]$ . From Remark 2.3 we see that **gu** holds for  $L_u > 2^*$  too, so we can construct  $W_{L_u}^{u,+}(\tau)$  and  $W_{L_u}^{u,-}(\tau)$ . Fix as above  $t_o \leq \mathfrak{T}$  and consider the flow of (S) for  $t \leq t_o$ . We construct again the triangle  $Z(\xi)$  and we observe that (3.10) and (3.11) continue to hold, and the flow of (S) on  $o(\xi)$  points outwards. So we conclude via Lemma A.6 as above the existence of a subset  $\mathcal{W}(\tau)$  of  $W_{L_u}^{u,+}(\tau)$  such that  $\mathbf{O} \in \mathcal{W}(\tau)$  and  $\mathcal{W}(\tau) \cap o(\xi) \neq \emptyset$ . So we prove the Lemma for  $W_{L_u}^{u,+}(\tau)$  for the arbitrariness of  $\xi$ . Then, recalling that  $W_{L_u}^{u,+}(\tau)$  and  $W_{l_u}^{u,+}(\tau)$  are omothetic we conclude.  $\square$

The following lemma investigates the presence of intersections between the unstable manifold  $W_{l_u}^{u,+}$  and the manifolds  $W_{l_u}^{s,\pm}$  which are omothetic to the stable manifold  $W_{l_s}^{s,\pm}$ . A similar reasoning was adopted already in [3, 13, 25, 30].

**Lemma 3.9.** *Assume the hypotheses of Theorem 2.15. Then  $W_{l_u}^{u,+}(\mathfrak{T})$  intersects  $W_{l_u}^s(\mathfrak{T})$  in a sequence of points  $\mathbf{Q}_j^{*,+}$ , for any  $j \in \mathbb{N}$ , where  $\mathfrak{T}$  is defined in **L1**. Moreover, we can assume that  $\mathbf{Q}_j^{*,+} \in W_{l_u}^{s,+}(\mathfrak{T})$  if  $j$  is even, while  $\mathbf{Q}_j^{*,+} \in W_{l_u}^{s,-}(\mathfrak{T})$  if  $j$  is odd.*

*Proof.* Fix  $\tau = \mathfrak{T}$ . From Lemma 3.7 we know that  $W_{l_u}^{s,+}(\tau)$  and  $W_{l_u}^{s,-}(\tau)$  are two spirals rotating counterclockwise around the origin, and each of them is cut infinitely many times by  $W_{l_u}^{u,+}(\tau)$  (see Figure 2): at least once at each rotation respectively at the point  $\mathbf{Q}_{2j}$  and  $\mathbf{Q}_{2j+1}$ , by the property shown in Lemma 3.8.

We develop this argument in polar coordinates too, for clarity and for later purposes. Using also Lemma 3.2 we see that

$$\begin{aligned} & \Theta^{s,+}(0, \tau) \in (-\pi/2, 0), \quad \Theta^{s,-}(0, \tau) \in (-3\pi/2, -\pi) \\ & \quad (\text{where } \Theta^{s,\pm}(0, \tau) := \lim_{L \rightarrow 0} \Theta^{s,\pm}(L, \tau)), \\ & \lim_{L \rightarrow L_\tau^\pm} \Theta^{s,\pm}(L, \tau) = +\infty, \\ & \Theta^{u,+}(d, \tau) \in (-\pi/2, \pi/2), \quad \Theta^{u,-}(d, \tau) \in (-3\pi/2, -\pi/2), \\ & \lim_{d \rightarrow d_\tau} R^{u,+}(d, \tau; l_u) = +\infty. \end{aligned}$$

Consider the following curves in the stripe  $(\Theta, R) \in \mathcal{S} = \mathbb{R} \times [0, +\infty)$ :

$$\begin{aligned}\Gamma^{u,\pm}(d, \tau) &:= (\Theta^{u,\pm}(d, \tau), R^{u,\pm}(d, \tau; l_u)), \\ \Gamma^{s,\pm}(L, \tau) &:= (\Theta^{s,\pm}(L, \tau), R^{s,\pm}(L, \tau; l_u)), \\ \Gamma_{2k}^s(L, \tau) &:= (\Theta^{s,+}(L, \tau) - 2k\pi, R^{s,+}(L, \tau; l_u)), \\ \Gamma_{2k+1}^s(L, \tau) &:= (\Theta^{s,-}(L, \tau) - 2k\pi, R^{s,-}(L, \tau; l_u)),\end{aligned}\tag{3.12}$$

for  $k \in \mathbb{N}$ . Let us introduce, for every  $k \geq 0$ ,

$$\begin{aligned}\hat{L}_{2k}^\uparrow &:= \min\{L > 0 \mid (\Theta^{s,+}(L, \tau) - 2k\pi = \pi/2)\}, \\ \hat{L}_{2k}^\downarrow &:= \min\{L > 0 \mid (\Theta^{s,+}(L, \tau) - 2k\pi = -\pi/2)\}, \\ \hat{L}_{2k+1}^\uparrow &:= \max\{L < 0 \mid (\Theta^{s,-}(L, \tau) - 2k\pi = \pi/2)\}, \\ \hat{L}_{2k+1}^\downarrow &:= \max\{L < 0 \mid (\Theta^{s,-}(L, \tau) - 2k\pi = -\pi/2)\},\end{aligned}\tag{3.13}$$

(except  $\hat{L}_0^\downarrow := 0$ ) and notice that  $|\hat{L}_j^\downarrow| < |\hat{L}_j^\uparrow| < |\hat{L}_{j+2}^\downarrow|$  holds by construction, all having the same sign. In fact, roughly speaking, we have denoted with “ $\uparrow$ ” the intersections of the stable manifold (the first at any lap) with the positive  $y$ -semiaxis and with “ $\downarrow$ ” the ones with the negative; moreover even subscripts correspond to the intersections of  $W_{l_u}^{s,+}(\tau)$ , respectively odd subscripts to the ones of  $W_{l_u}^{s,-}(\tau)$  (cf. Figure 2a).

Let  $\hat{F}_j^\uparrow(\tau)$  be the open region delimited by  $\{\Omega = \Gamma_j^s(L, \tau) \mid 0 \leq |L| \leq |\hat{L}_j^\uparrow|\}$  and the lines  $R = 0$  and  $\Theta = \pi/2$  (see again Figure 2a). Similarly let  $\hat{F}_j^\downarrow(\tau)$  be the open region delimited by  $\{\Omega = \Gamma_j^s(L, \tau) \mid 0 \leq |L| \leq |\hat{L}_j^\downarrow|\}$  and the lines  $R = 0$  and  $\Theta = -\pi/2$ . It is easy to check (see e.g. the proof of Lemma 3.9 in [13] for details) that, for any  $j \in \mathbb{N}$ , the curve  $d \rightarrow \Gamma^{u,+}(\cdot, \tau)$  is in  $\hat{F}_j^\uparrow(\tau)$  for  $d$  small and outside for  $d$  large. So it intersects the graph of  $\Gamma_j^s$ : these intersections corresponds to distinct points  $\mathbf{Q}_j \in \mathbb{R}^2$ . We are interested in the first intersections, in the sense of the parameter  $d$ , so let us set

$$\begin{aligned}d_0^* &:= \min\{d > 0 \mid \exists L \in (0, L_\tau^+) : \Gamma^{u,+}(d, \tau) = \Gamma_0^s(L, \tau)\}, \\ d_1^* &:= \min\{d > d_0^* \mid \exists L \in (L_\tau^-, 0) : \Gamma^{u,+}(d, \tau) = \Gamma_1^s(L, \tau)\}, \\ d_{2k}^* &:= \min\{d > d_{2k-1}^* \mid \exists L \in (0, L_\tau^+) : \Gamma^{u,+}(d, \tau) = \Gamma_{2k}^s(L, \tau)\}, \\ d_{2k+1}^* &:= \min\{d > d_{2k}^* \mid \exists L \in (L_\tau^-, 0) : \Gamma^{u,+}(d, \tau) = \Gamma_{2k+1}^s(L, \tau)\},\end{aligned}\tag{3.14}$$

for any  $k \geq 1$ . We denote by  $L_j^*$  the unique value in  $(L_\tau^-, L_\tau^+)$  such that  $\Gamma^{u,+}(d_j^*, \tau) = \Gamma_j^s(L_j^*, \tau)$ , and we set  $\mathbf{Q}_j^{*,+}(\tau) := \Upsilon_{l_u}^{u,+}(d_j^*, \tau)$ , and  $\Omega_j^*(\tau) := \Gamma^{u,+}(d_j^*, \tau)$ , the corresponding *switched* polar coordinates (see Figure 2a). Moreover, notice that by construction  $|\hat{L}_j^\downarrow| < |L_j^*| < |\hat{L}_j^\uparrow|$  holds, all having the same sign.  $\square$

**Remark 3.10.** A similar result can be obtained for the unstable manifold  $W_{l_u}^{u,-}(\tau)$  (cf. Figure 2a): the curve  $\Gamma^{u,-}(\cdot, \tau)$  must exit from  $\hat{F}_j^\downarrow(\tau)$ , so  $W_{l_u}^{u,-}(\tau)$  intersects the stable manifold  $W_{l_s}^s(\tau)$  in the distinct points  $\mathbf{Q}_j^{*, -}(\tau)$  with switched polar coordinates  $\Omega_j^{*, -}(\tau) \in \mathcal{S}$ . In particular  $\mathbf{Q}_j^{*, -}(\tau) \in W_{l_u}^{s,-}(\tau)$  if  $j$  is even,



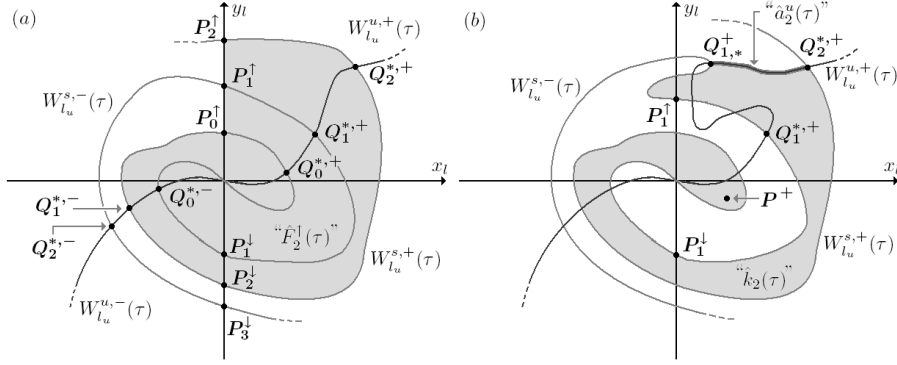


Figure 2: We give here a sketch of the introduced notations. For simplicity we represent the sets not on the stripe  $\mathcal{S}$  but in the  $(x_l, y_l)$ -plane. The points  $P_i^\uparrow$  and  $P_i^\downarrow$  correspond to the coordinate values  $\hat{L}_i^\uparrow$  and  $\hat{L}_i^\downarrow$  introduced in (3.13). In (a) we have colored the region “ $\hat{F}_2^\uparrow(\tau)$ ” corresponding to  $\hat{F}_2^\uparrow(\tau)$  for illustrative purpose: being  $W_{l_u}^{u,+}(\tau)$  unbounded, then it exits from “ $\hat{F}_2^\uparrow(\tau)$ ” for the first time at  $Q_2^{*,+} = \Upsilon_{l_u}^{u,+}(d_2^*, \tau) = \Upsilon_{l_u}^{s,+}(L_2^*, \tau)$ . So, we can verify that  $|\hat{L}_2^\downarrow| < |L_2^*| < |\hat{L}_2^\uparrow|$  holds. Similarly  $W_{l_u}^{s,-}(\tau)$  exits from “ $\hat{F}_2^\uparrow(\tau)$ ” for the first time at  $Q_1^{*, -} = \Upsilon_{l_u}^{s,-}(d_1^{*, -}, \tau) = \Upsilon_{l_u}^{s,+}(L_1^{*, -}, \tau)$  and  $|\hat{L}_0^\uparrow| < |L_1^{*, -}| < |\hat{L}_2^\downarrow|$  is verified. In (b) the more tricky situation described in Lemma 3.12 is drawn:  $W_{l_u}^{u,+}(\tau)$  exits from “ $\hat{F}_1^\uparrow(\tau)$ ” (not colored) for the first time at  $Q_1^{*,+} = \Upsilon_{l_u}^{u,+}(d_1^*, \tau)$ , then it exits from “ $\hat{F}_2^\uparrow(\tau)$ ” for the first time at  $Q_2^{*,+} = \Upsilon_{l_u}^{u,+}(d_2^*, \tau)$ . Between these points we locate  $Q_{1,*}^+ = \Upsilon_{l_u}^{u,+}(d_{*,1}, \tau)$  (so that  $d_{*,1} > d_1^*$ ). We have highlighted the set corresponding to  $\hat{a}_2^u(\tau)$  defined in (3.16) and colored the region corresponding to  $\hat{k}_2(\tau)$ . We emphasize that in the simpler case  $d_1^* = d_{*,1}$ ,  $\hat{a}_2^u(\tau)$  consists of the whole branch between  $Q_1^{*,+}$  and  $Q_2^{*,+}$  as in (a). Finally, arguing as in the proof of Remark 3.15 all the points of  $\hat{a}_2^u(\tau)$  and  $\hat{k}_2(\tau)$  tend to  $P^+$  as  $t \rightarrow +\infty$ .

while  $Q_j^{*, -}(\tau) \in W_{l_u}^{s,+}(\tau)$  if  $j$  is odd. Moreover, we can identify the corresponding sequences  $(d_j^{*, -})_{j \geq 0}$ ,  $(L_j^{*, -})_{j \geq 0}$ , with  $d_j^{*, -} < 0$  and  $(-1)^j L_j^{*, -} < 0$ , such that  $\Omega_j^{*, -}(\tau) = \Gamma_{l_u}^{u,-}(d_j^{*, -}, \tau) = \Gamma_j^s(L_j^{*, -}, \tau)$ . Let us set  $\hat{L}_{-1}^\uparrow := 0$ ; by construction we find  $|\hat{L}_{j-1}^\uparrow| < |L_j^{*, -}| < |\hat{L}_{j+1}^\downarrow|$ , all having the same sign. We stress that the sequences in the statement of Theorem 2.15 can be now introduced as  $A_j^+ = d_j^*$ ,  $A_j^- = |d_j^{*, -}|$ ,  $B_{2k}^+ = L_{2k}^*$  and  $B_{2k+1}^+ = L_{2k+1}^{*, -}$ ,  $B_{2k}^- = |L_{2k}^{*, -}|$  and  $B_{2k+1}^- = |L_{2k+1}^{*, -}|$ : they are strictly increasing by construction.

By construction,  $\mathbf{x}_{l_u}(t; \mathfrak{T}, Q_j^{*,+}(\mathfrak{T}))$  is a homoclinic trajectory of (S), and the corresponding solution  $u(r, d_j^*)$  of (Hr) is a  $\mathcal{R}$ -fd solution. We introduce the following notation: let  $\bar{\Omega} = (\bar{\Theta}, \bar{R})$  be a point in  $\mathcal{S}$ , and  $\bar{Q} = \bar{R}(\cos(\bar{\Theta}), \sin(\bar{\Theta}))$ ; we denote by  $\Omega_{l_u}(t, \mathfrak{T}, \bar{\Omega}) = (\phi(t, \mathfrak{T}, \bar{\Omega}), \rho_{l_u}(t, \mathfrak{T}, \bar{\Omega}))$  the switched polar coordinates of  $\mathbf{x}_{l_u}(t, \mathfrak{T}, \bar{Q})$ , such that  $\Omega_{l_u}(\mathfrak{T}, \mathfrak{T}, \bar{\Omega}) = \bar{\Omega}$  (so that it is uniquely defined). With

a little abuse of notation we denote by  $\Omega_{l_s}(t, \mathfrak{T}, \bar{\Omega}) = (\phi(t, \mathfrak{T}, \bar{\Omega}), \rho_{l_s}(t, \mathfrak{T}, \bar{\Omega}))$  the switched polar coordinates of  $\mathbf{x}_{l_s}(t, \mathfrak{T}, \bar{\mathbf{R}})$ , where  $\bar{\mathbf{R}} = \bar{\mathbf{Q}}e^{(\alpha_{l_s} - \alpha_{l_u})\mathfrak{T}}$ , and we observe that the angular coordinate is the same as in  $\Omega_{l_u}(t, \mathfrak{T}, \bar{\Omega})$ , while the radial coordinate is multiplied by  $e^{(\alpha_{l_s} - \alpha_{l_u})t}$ .

**Lemma 3.11.** *Assume the hypotheses of Theorem 2.15. Then,  $u(r, d_j^*)$  is  $\mathcal{R}^{\downarrow}\text{fd}$ . In particular,  $u(r, d_0^*)$  is a positive solution.*

*Proof.* When we consider (Lr) we can simply repeat the proof of [13, Lemma 3.11] with no changes. When we deal with (Hr) we need to adapt slightly the argument: we sketch the proof for reader's convenience.

If  $j = 2k$  is even we have  $\phi(\mathfrak{T}, \mathfrak{T}, \Omega_{2k}^*) = \Theta^{u,+}(d_{2k}^*, \mathfrak{T}) = \Theta^{s,+}(L_{2k}^*, \mathfrak{T}) - 2k\pi$ , while if  $j = 2k + 1$  is odd we have  $\phi(\mathfrak{T}, \mathfrak{T}, \Omega_{2k+1}^*) = \Theta^{u,+}(d_{2k+1}^*, \mathfrak{T}) = \Theta^{s,-}(L_{2k+1}^*, \mathfrak{T}) - 2k\pi$ , see also Lemma 3.6. Therefore  $\mathbf{x}_{l_u}(t, \mathfrak{T}, \mathbf{Q}_j^*)$  performs the angle  $\phi(\mathfrak{T}, \mathfrak{T}, \Omega_j^*) + \arctan(\kappa(\eta))$  around the origin when  $t \in (-\infty, \mathfrak{T}]$ , and it performs the angle  $-j\pi - \arctan(n - 2 - \kappa(\beta)) - \phi(\mathfrak{T}, \mathfrak{T}, \Omega_j^*)$  for every  $j$  when  $t \in [\mathfrak{T}, +\infty)$ . Summing up,  $\mathbf{x}_{l_u}(t, \mathfrak{T}, \mathbf{Q}_j^*)$  performs the angle  $-j\pi - \arctan(n - 2 - \kappa(\beta)) + \arctan(\kappa(\eta))$ . Since  $\arctan(n - 2 - \kappa(\beta)) - \arctan(\kappa(\eta)) \in (0, \pi)$  we see that  $\mathbf{x}_{l_u}(t, \mathfrak{T}, \mathbf{Q}_j^*)$  crosses the  $y$  axis exactly  $j$  times and so  $u(r, d_j^*)$  changes sign exactly  $j$  times.  $\square$

**Remark 3.12.** *The solution  $u(d_j^*, r)$  of (Hr) is a  $\mathcal{R}^{\downarrow}\text{fd}$ , and it is definitively positive for  $j$  even and definitively negative for  $j$  odd. However, a priori, we may find some  $d > 0$ ,  $d \neq d_k^*$  for any  $k \geq 0$  such that  $u(d, r)$  is a  $\mathcal{R}^{\downarrow}\text{fd}$  (cf. Figure 2b). In fact it may happen e.g. that  $d \rightarrow \Gamma^{u,+}(d, \mathfrak{T})$  intersects  $\Gamma_0^s(\cdot, \mathfrak{T})$  in, say, three points: at  $d = d_0^* < d_a < d_b$ , thus corresponding to three distinct points  $\mathbf{Q}_0^*, \mathbf{Q}_0^a, \mathbf{Q}_0^b \in \mathbb{R}^2$ . In this case we have three positive  $\mathcal{R}^{\downarrow}\text{fd}$ :  $u(r, d_0^*)$ ,  $u(r, d_0^a)$ ,  $u(r, d_0^b)$ , see [13, Section 3] for a more detailed discussion of this point.*

**Remark 3.13.** *We emphasize that  $\Gamma^{u,+}(d, \tau)$  and  $\Gamma_{2k}^s(L, \tau)$  are well defined for any  $0 \leq d < d_{\mathfrak{T}}^*$ ,  $0 \leq L < L_{\mathfrak{T}}^*$ , whenever  $\tau \geq \mathfrak{T}$ , cf L1. Next if  $\Gamma^{u,+}(\cdot, \mathfrak{T})$  intersects  $\Gamma_j^s(\cdot, \mathfrak{T})$  in  $\Omega_j^*$ , then  $\Gamma^{u,+}(\cdot, \tau)$  intersects  $\Gamma_j^s(\cdot, \tau)$  in a point  $\Omega_j^*(\tau)$  corresponding to the same solution  $u(d_j^*, r)$ , for any  $\tau \geq \mathfrak{T}$ .*

Following the ideas of [13, Section 3], let us now turn to consider the solution  $u(r, d)$  where  $d \neq d_j^*$  for any  $j \in \mathbb{N}$ . Fix  $\tau \geq \mathfrak{T}$ , we need to define several subsets of the stripe  $\mathcal{S}$ :

$$\begin{aligned} \hat{A}_j^u(\tau) &:= \{\Omega = \Gamma^u(d, \tau) \mid 0 \leq d \leq d_j^*\}, \\ \hat{B}_{2k}^s(\tau) &:= \{\Omega = \Gamma_{2k}^s(L, \tau) \mid 0 \leq L \leq L_{2k}^*\}, \\ \hat{B}_{2k+1}^s(\tau) &:= \{\Omega = \Gamma_{2k+1}^s(L, \tau) \mid L_{2k+1}^* \leq L \leq 0\}, \end{aligned} \quad (3.15)$$

for any  $j, k \in \mathbb{N}$ . We denote by  $\hat{E}_j(\tau)$  the open set enclosed by  $\hat{A}_j^u(\tau)$ ,  $\hat{B}_j^s(\tau)$  and the line  $R = 0$ , which is bounded for any  $\tau \geq \mathfrak{T}$  and any  $j \in \mathbb{N}$ . Further set  $d_{*, -1} := 0$  and, for  $j \geq 0$ ,

$$d_{*, j} := \max\{d \in [0, d_{j+1}^*) \mid \text{there is } L \in \mathbb{R} \text{ so that } \Gamma^{u,+}(d, \tau) = \Gamma_j^s(L, \tau)\}.$$

Observe that if  $\Gamma^{u,+}(\cdot, \tau)$  has a unique intersection with  $\Gamma_j^s(\cdot, \tau)$  then  $d_{*,j} = d_j^*$  (in general we have  $d_{*,j} \geq d_j^*$ ). We set (cf. Figure 2b)

$$\hat{a}_j^u(\tau) := \{\Omega = \Gamma^u(d, \tau) \mid d_{*,j-1} < d < d_j^*\}. \quad (3.16)$$

Define  $\hat{k}_0(\tau) = \hat{E}_0(\tau)$  and, for any  $j > 0$ , denote by  $\hat{k}_j(\tau)$  the open bounded set enclosed by  $\hat{B}_j^s(\tau)$ ,  $\hat{a}_j^u(\tau)$ ,  $\hat{B}_{j-1}^s(\tau)$  and the line  $R = 0$  (observe that  $\hat{k}_j(\tau) \subset \hat{E}_j(\tau)$ ). Note that these sets have the following property.

**Remark 3.14.** *If  $\bar{\Omega}$  belongs to  $\hat{A}_j^u(\tau)$ ,  $\hat{B}_j^s(\tau)$ ,  $\hat{E}_j(\tau)$ ,  $\hat{k}_j(\tau)$ , for some  $\tau \geq \mathfrak{T}$  then  $\Omega(t, \tau, \bar{\Omega})$  belongs respectively to  $\hat{A}_j^u(t)$ ,  $\hat{B}_j^s(t)$ ,  $\hat{E}_j(t)$ ,  $\hat{k}_j(t)$  for any  $t \geq \mathfrak{T}$ .*

*Proof.* We just sketch the proof which is strongly inspired by [13, Section 3] and in particular [13, Lemma 3.14]. The claim concerning  $\hat{A}_j^u(\tau)$ ,  $\hat{B}_j^s(\tau)$  follows from construction. Then note that if  $\bar{\Omega} \notin \hat{A}_j^u(\tau)$  then  $\Omega(t, \tau, \bar{\Omega}) \notin \hat{A}_j^u(t)$  too, and the same property holds for  $\hat{B}_j^s(\tau)$ . It follows that if  $\bar{\Omega} \in \hat{E}_j(\tau)$  then  $\Omega(t, \tau, \bar{\Omega})$  cannot cross  $\hat{A}_j^u(t)$  and  $\hat{B}_j^s(t)$ , hence it is forced to stay in  $\hat{E}_j(t)$  for any  $t \in \mathbb{R}$ . The claim concerning  $\hat{k}_j(\tau)$  is analogous.  $\square$

Now we turn to consider  $\mathcal{R}_{\text{sd}}$  solutions.

**Lemma 3.15.** *Assume the hypotheses of Theorem 2.15 and fix  $\tau \geq \mathfrak{T}$ . Then solutions  $u(r, d)$  of (Hr) corresponding to  $\Omega(t, \tau, \bar{\Omega})$  with  $\bar{\Omega} \in \hat{a}_j^u(\tau)$  are  $\mathcal{R}_{\text{sd}}$ . In particular,  $u(r, d)$  is a positive solution, for any  $0 < d < d_0^*$ .*

*Proof.* From Lemma 3.8, we know that  $\hat{A}_j^u(\mathfrak{T}) \in \{(\Theta, R) \mid |\Theta| < \frac{\pi}{2}\}$ . Moreover  $\hat{A}_j^u(\tau)$  is a path connecting the point  $\Omega_a(\tau) = (\Theta^u(\tau), 0)$  to  $\Omega_j^{*,+}(\tau)$ , where  $\Theta^u(\tau) = \arctan(m^u(\tau)) \in (-\arctan \frac{n-2}{2}, \frac{\pi}{2})$ . From Lemma 3.7 we see that  $\hat{B}_j^s(\tau)$  is a path connecting a point  $\Omega_b^j(\tau) = (\Theta^j(\tau), 0)$  to  $\Omega_j^{*,+}(\tau)$ , where  $\Theta^j(\tau) + j\pi = \arctan(m^s(\tau)) \rightarrow -\arctan(n-2-\kappa(\beta))$  as  $\tau \rightarrow +\infty$ , and  $|\Theta^j(\tau) + j\pi| < \frac{\pi}{2}$ .

Let  $\hat{\Omega}(t) = (\phi(t), \hat{\rho}(t)) = \Omega(t, \tau, \bar{\Omega})$  be the switched polar coordinates of  $\mathbf{x}_{\mathbf{l}_u}(t, \tau, \mathbf{Q})$  and  $\tilde{\Omega}(t) = (\phi(t), \tilde{\rho}(t))$  the switched polar coordinates of the trajectory  $\mathbf{x}_{\mathbf{l}_s}(t, \tau, \mathbf{Q}e^{(\alpha_{\mathbf{l}_s} - \alpha_{\mathbf{l}_u})t})$  corresponding to the same solution  $u(r)$  of (Hr), so that  $\tilde{\rho}(t) = \hat{\rho}(t)e^{(\alpha_{\mathbf{l}_s} - \alpha_{\mathbf{l}_u})t}$ .

Let us denote by  $\tilde{A}_j^u(\tau) := \{(\Theta, R e^{(\alpha_{\mathbf{l}_s} - \alpha_{\mathbf{l}_u})\tau}) \mid (\Theta, R) \in \hat{A}_j^u(\tau)\}$ ,  $\tilde{B}_j^s(\tau) := \{(\Theta, R e^{(\alpha_{\mathbf{l}_s} - \alpha_{\mathbf{l}_u})\tau}) \mid (\Theta, R) \in \hat{B}_j^s(\tau)\}$ , and similarly for  $\tilde{a}_j^u(\tau)$ ,  $\tilde{E}_j(\tau)$ ,  $\tilde{k}_j(\tau)$ . By construction  $\hat{\Omega}(t) \in \hat{A}_j^u(t)$ ,  $\hat{B}_j^s(t)$ ,  $\hat{a}_j^u(t)$ ,  $\hat{E}_j(t)$ ,  $\hat{k}_j(t)$ , iff  $\tilde{\Omega}(t) \in \tilde{A}_j^u(t)$ ,  $\tilde{B}_j^s(t)$ ,  $\tilde{a}_j^u(t)$ ,  $\tilde{E}_j(t)$ ,  $\tilde{k}_j(t)$ .

Denote by  $\tilde{\Gamma}_{l_s}^{s,\pm}(\sigma, +\infty)$  the switched polar coordinates for  $\Sigma_{l_s}^{s,\pm}(\sigma, +\infty)$ , and set

$$\begin{aligned} \tilde{\Gamma}_{2k, l_s}^s(\sigma, +\infty) &= \{\tilde{\Omega} - (2k\pi, 0) \mid \tilde{\Omega} \in \tilde{\Gamma}_{l_s}^{s,+}(\sigma, +\infty)\}, \\ \tilde{\Gamma}_{2k+1, l_s}^s(\sigma, +\infty) &= \{\tilde{\Omega} - (2k\pi, 0) \mid \tilde{\Omega} \in \tilde{\Gamma}_{l_s}^{s,-}(\sigma, +\infty)\}. \end{aligned}$$

We define  $\tilde{K}_j(+\infty)$  as the unbounded open stripe between  $\tilde{\Gamma}_{j, l_s}^s(\sigma, +\infty)$  and  $\tilde{\Gamma}_{j-1, l_s}^s(\sigma, +\infty)$ . Observe further that  $\tilde{K}_j(+\infty)$  corresponds to the unbounded

open subset between  $W_{l_s}^{s,+}(+\infty)$  and  $W_{l_s}^{s,-}(+\infty)$  for (S), the one containing  $\mathbf{P}^+$  if  $j$  is even and the one the one containing  $\mathbf{P}^-$  if  $j$  is odd.

Note that  $\tilde{k}_j(\tau)$  as  $\tau \rightarrow +\infty$  approaches a bounded open subset of  $\tilde{K}_j(+\infty)$ , say  $\tilde{k}_j(+\infty)$  containing the switched polar coordinates either of  $\mathbf{P}^+$  if  $j$  is even or of  $\mathbf{P}^-$  if  $j$  is odd; similarly  $\tilde{a}_j^u(\tau)$  approaches a subset of  $\tilde{K}_j(+\infty)$  as  $\tau \rightarrow +\infty$ . Moreover note that  $\mathbf{P}^+$  (or  $\mathbf{P}^-$ ) is the unique attracting subset of  $\tilde{k}_j(+\infty)$ . Let  $(\Theta^\pm, R^\pm)$  be switched polar coordinates for  $\mathbf{P}^\pm$ .

Then observe that if  $\tilde{\Omega} \in \hat{k}_{2k}(\tau)$  (respectively  $\tilde{\Omega} \in \hat{k}_{2k+1}(\tau)$ ) then  $\tilde{\Omega}(\tau) \in \tilde{k}_{2k}(\tau)$  (resp.  $\tilde{\Omega}(\tau) \in \tilde{k}_{2k+1}(\tau)$ ) and  $\tilde{\Omega}(t)$  converges to  $(\Theta^+ - 2k\pi, R^+)$  (resp.  $(\Theta^- - 2k\pi, R^-)$ ) as  $t \rightarrow +\infty$ . Moreover if  $\tilde{\Omega} \in \hat{a}_{2k}^u(\tau)$  (respectively  $\tilde{\Omega} \in \hat{a}_{2k+1}^u(\tau)$ ) then  $\tilde{\Omega}(t) \rightarrow (-\arctan(\kappa(\eta)), 0)$  as  $t \rightarrow -\infty$ , and again  $\tilde{\Omega}(\tau) \in \tilde{a}_{2k}^u(\tau)$  (resp.  $\tilde{\Omega}(\tau) \in \tilde{a}_{2k+1}^u(\tau)$ ), and  $\tilde{\Omega}(t)$  converges to  $(\Theta^+ - 2k\pi, R^+)$  (resp.  $(\Theta^- - 2k\pi, R^-)$ ) as  $t \rightarrow +\infty$ ; hence the corresponding solution  $u(r)$  of (Hr) is a  $\mathcal{R}^{2k}\text{sd}$  (resp. a  $\mathcal{R}^{2k+1}\text{sd}$ ) which is definitively positive (resp. negative).  $\square$

We stress that by construction we find  $a_j^+ = d_{*,j-1}$ , where  $(a_k^+)_{k \geq 1}$  is the sequence in the statement of Theorem 2.15. Lemmas 3.11 and 3.15 can be reformulated also for  $\mathcal{R}$ -solutions  $u(d, r)$  with  $d < 0$ , reasoning similarly on the curve  $\Gamma^{u,-}(\cdot, \tau)$ . So the proof of Theorem 2.15 is concluded.

## 4 Some further examples.

In this section we briefly present other types of nonlinearities to which our theorems apply. Let us begin by noticing that when  $f$  is as in (1.2a), arguing as in Section 2.1, we can choose  $l_u$  and  $l_s$  as in Corollary 1.2, and we get

$$g_{l_u}(x, t) = [K(0) + \Delta_u(-t)]|x|^{q-2}x, \quad g_{l_s}(x, t) = [K(\infty) + \Delta_s(t)]|x|^{q-2}x,$$

where  $K(0) < 0 < K(\infty)$ , and  $\Delta_u(T)$  and  $\Delta_s(T)$  go to 0 as  $T \rightarrow +\infty$ . In fact we can also consider logarithmic growth, e.g.

$$K(r) \sim K(0)|\ln(r)|^{a_0}r^{\delta_0} \quad \text{as } r \rightarrow 0, \quad (4.1)$$

where  $K(0) < 0$ ,  $a_0 \in \mathbb{R}$ ,  $\delta_0 > -2$ . In this case assumption **Gu** does not hold but **gu** holds for any  $l_u \in (l(q, \delta_0), l(\eta))$ , see (1.7). So if **K** holds with the first in (1.6) replaced by (4.1) we can apply Corollary 1.2. Similarly if  $f$  is as in (1.2b) and **K** holds with the second in (1.6) replaced by  $K(r) \sim K(\infty)|\ln(r)|^{a_\infty}r^{\delta_\infty}$ ,  $K(\infty) > 0$ , then **gs** holds for any  $l_s \in (2_*(\beta), l(q, \delta_\infty))$ , so we can apply Corollary 1.3.

Introduce

$$f_i(u, r) = \sum_{j=1}^{N_i} K_{i,j}(r)r^{\delta_{i,j}}|u|^{q_{i,j}-2}u \quad (4.2)$$

for a certain integer  $N_i$ , where  $\delta_{i,j} > -2$ ,  $q_{i,j} > 2$ ,  $K_{i,j}$  are continuous functions which are bounded and uniformly far from zero for  $r$  small and  $r$  large, negative

near zero, changes sign at  $R_{i,j} > 0$  and then they are positive. We assume

$$\ell_i = \ell_{i,j} = l(q_{i,j}, \delta_{i,j}), \quad \text{for every } j,$$

thus having

$$g_{l_i}(x, t) = \sum_{j=1}^{N_i} K_{i,j}(e^t) |x|^{q_{i,j}-2} x.$$

Moreover it is possible to assume that some of the  $K_{i,j}$  are identically zero for either  $r \leq R_{i,j}$  or  $r \geq R_{i,j}$ . We do to not enter in details for major clarity. Assume now

$$f(u, r) = \sum_{i=1}^m f_i(u, r), \quad \ell_1 > \ell_2 > \cdots > \ell_{m-1} > \ell_m. \quad (4.3)$$

Setting  $l_u = \ell_1$  and remembering that, for  $L \neq l$ , one has  $g_L(x, t)/x = g_l(\xi, t)/\xi$ , where  $\xi = xe^{(\alpha_l - \alpha_L)t}$ , a computation gives the validity of **gu** if  $l_u < I(\eta)$ . The first condition of **L1** holds simply setting  $\mathfrak{T} = \min\{R_{i,j}\}$ , while we have to assume

$$\text{if } R_{i_o, j_o} = \mathfrak{T} \quad \text{then} \quad q_{i_o, j_o} \geq q_{i,j} \text{ for every } (i, j) \neq (i_o, j_o) \quad (4.4)$$

in order to fulfill the second one. Roughly speaking, the term with the greatest power is the first to change sign.

Assumption **Gs** holds setting  $l_s = \ell_m$ . Hence, Theorem 2.15 applies if  $f$  is as in (4.3) with  $2^* < \ell_m < \cdots < \ell_1 < I(\eta)$  and such that (4.4) holds.

Arguing as above we can find the corresponding specular conditions in order to permit the application of Theorem 2.16. Assume  $f(u, r) = -\sum_{i=1}^m f_i(u, r)$  with  $f_i$  as in (4.2) assuming now  $\ell_1 < \ell_2 < \cdots < \ell_{m-1} < \ell_m$ . Setting  $l_s = \ell_1 > 2_*(\beta)$  we have **gs**, then set  $\mathfrak{T} = \max\{R_{i,j}\}$  and assume (4.4) so that **L2** is given. The validity of **Gu** is given setting  $l_u = \ell_m$ , asking  $2_*(\eta) < \ell_m < 2^*$ .

The functions previously considered consist of sum of possibly different polynomial terms. However, our results permit us to consider also more general nonlinearities, which however have a leading term in their expansion for  $u$  small and  $u$  large which is polynomial, e.g.

$$f(u, r) = \frac{|u|^{q_1-2}u}{1+u^{q_2}} \cdot \frac{r^{\delta_1}}{1+r^{\delta_2}}.$$

assuming  $q_1 > 2$ ,  $q_1 - q_2 > 2$  and  $\delta_1 > -2$ ,  $\delta_1 - \delta_2 > -2$ . In such a case, a straightforward computation gives that **Gu** holds requiring  $l_u = l(q_1 - q_2, \delta_1)$  and **Gs** holds setting  $l_s = l(q_1, \delta_1 - \delta_2)$ .

Further notice that our results are robust. I.e., we have the following.

**Remark 4.1.** Assume **H** and consider (for simplicity) the functions  $\tilde{f}^+(u, r)$ ,  $\tilde{f}^-(u, r)$  satisfying the assumption of Corollary 1.2, and 1.3 respectively.

Let  $\bar{f}(u, r)$  be such that  $\bar{f}(0, r) = \frac{\partial \bar{f}}{\partial u}(0, r) = 0$ ; suppose that there is  $C > 1$  such that  $\bar{f}(u, r) \equiv 0$  for  $r < 1/C$  and for  $r > 1/C$ , and that  $\lim_{|u| \rightarrow +\infty} \frac{\bar{f}(u, r)}{|u|^{q-1}} = 0$  uniformly for  $r > 0$ .

Then the function  $f(u, r) = \tilde{f}^+(u, r) + \bar{f}(u, r)$  satisfies the assumptions of Theorem 2.15, and the solutions of (Hr) have the structure described in Corollary 1.2 (and Theorem 2.15).

Similarly the function  $f(u, r) = \tilde{f}^-(u, r) + \bar{f}(u, r)$  satisfies the assumptions of Theorem 2.16, and the solutions of (Hr) have the structure described in Corollary 1.3 (and Theorem 2.16).

The assumptions on Hardy potentials  $h(r)$  are more clear so we just emphasize the following interesting example satisfying **H**:

$$u'' + \frac{n-1}{r}u' + \frac{C}{1+r^2}u + f(u, r) = 0, \quad \text{with } C < \frac{(n-2)^2}{4},$$

$$\text{i.e. } h(r) = \frac{Cr^2}{1+r^2}, \quad \text{with } \eta = 0, \text{ and } \beta = C.$$

## A Appendix

### A.1 On the lack of continuability

In this appendix we first review briefly some well known facts concerning exponential dichotomy, see, e.g., [12]. Then we develop the construction of stable and unstable manifolds for non-autonomous systems, i.e.  $W_{l_u}^u(\tau)$  and  $W_{l_s}^s(\tau)$ , when continuability of the trajectories of (S) is ensured, i.e. when hypothesis **C** holds. Then we extend our discussion to the case where **C** does not hold.

Denote by  $\mathcal{A}_l(t) = \begin{pmatrix} \alpha_l & 1 \\ -h(e^t) & \gamma_l \end{pmatrix}$  the linearization of the right hand side of (S) in the origin, and by  $\mathcal{A}_l(\pm\infty) = \lim_{t \rightarrow \pm\infty} \mathcal{A}_l(t)$ . Assume either **Gu** or **gu**: note that  $\mathcal{A}_{l_u}(-\infty)$  has  $\lambda_2 < 0 < \lambda_1$  as eigenvalues where  $\lambda_1 := \alpha_{l_u} - \kappa(\eta)$  and  $\lambda_2 := \alpha_{l_u} + 2 - n + \kappa(\eta)$ . By **H**,  $\mathcal{A}_{l_u}(t)$  can be seen as an  $L^1$  perturbation of  $\mathcal{A}_{l_u}(-\infty)$ , therefore it admits exponential dichotomy in negative semi-lines  $(-\infty, \tau]$ . More precisely let  $X(t)$  be the fundamental matrix of

$$\dot{x} = \mathcal{A}_{l_u}(t)x, \tag{A.1}$$

i.e. the matrix solution of (A.1) such that  $X(0) = I$ , where  $I$  denotes the identity matrix. Then, for any  $\tau \in \mathbb{R}$  there is a constant  $K = K(\tau) > 1$ , exponents  $\bar{\lambda}_2 < 0 < \bar{\lambda}_1$  and a projection  $\mathcal{P}^-$  such that

$$\begin{aligned} \|X(t)(I - \mathcal{P}^-)X(s)^{-1}\| &\leq Ke^{\bar{\lambda}_1(t-s)} & \text{for any } t < s < \tau, \\ \|X(t)\mathcal{P}^-X(s)^{-1}\| &\leq Ke^{\bar{\lambda}_2(t-s)} & \text{for any } s < t < \tau, \end{aligned} \tag{A.2}$$

see, e.g., [12, Section 4]. Moreover the optimal choice for  $\bar{\lambda}_i$  is  $\bar{\lambda}_i = \lambda_i$  for  $i = 1, 2$ , see [8, Appendix]. Let us denote by  $\mathcal{P}^-(\tau) := X(\tau)\mathcal{P}^-X(\tau)^{-1}$ , and by  $\ell^u(\tau)$  the 1-dimensional kernel of  $\mathcal{P}^-(\tau)$ ; then  $\ell^u(\tau)$  is the unstable space

for (A.1). I.e let  $\vec{\xi} \in \mathbb{R}^2$ , and denote by  $\vec{\xi}(t)$  the solution of (A.1) such that  $\vec{\xi}(\tau) = \vec{\xi}$ ; then  $\vec{\xi}(t)$  is bounded for  $t \leq 0$  iff  $\vec{\xi} \in \ell^u(\tau)$ , cf. [12, Section 4]. Since  $\ell^u(\tau)$  is 1-dimensional we see that there is  $c = c(\vec{\xi})$  such that  $\|\vec{\xi}(t)\|e^{-\lambda_1 t} \rightarrow c$  as  $t \rightarrow -\infty$ . Also note that by construction  $\ell^u(\tau)$  is a line, and  $\vec{\xi} \in \ell^u(\tau)$  iff  $\vec{\xi}(t) \in \ell^u(t)$ .

Now assume **gu** and consider (S) where  $l = l_u$ : we consider this problem as a nonlinear perturbation of (A.1). Thus, setting  $Q(\delta) = \{(x, y) \mid |x| \leq \delta, |y| \leq \delta\}$ , we get the following, see [27, Theorem 2.25].

**Lemma A.1.** *Assume **gu**; then for any  $N \in \mathbb{R}$  we can find  $\delta = \delta(N)$  such that the set*

$$W_{l_u, loc}^u(\tau) := \left\{ \mathbf{Q} \in Q(\delta) \mid \mathbf{x}_{l_u}(t, \tau; \mathbf{Q}) \in Q(\delta) \text{ for any } t \leq \tau, \right. \\ \left. \text{and } \lim_{t \rightarrow -\infty} \mathbf{x}_{l_u}(t, \tau; \mathbf{Q}) = (0, 0) \right\} \quad (\text{A.3})$$

*is a graph on  $\ell^u(\tau) \cap Q(\delta)$  for any  $\tau \leq N$ . Moreover  $\ell^u(\tau)$  is the tangent space to  $W_{l_u, loc}^u(\tau)$  in the origin.*

We sketch the proof for completeness. Assume **gu** and suppose first that,

$$|g_{l_u}(x_2, t) - g_{l_u}(x_1, t)| \leq c(\tau)|x_2 - x_1|, \quad \text{for any } t \leq \tau$$

for some  $c(\tau) > 0$  and for any  $x_1, x_2 \in \mathbb{R}$ . Then, using a variation of constants formula, see e.g. [11, Section 3.3] or [27, Theorem 2.25], we prove that the set, cf. (2.6),

$$\tilde{W}_{l_u}^u(\tau) := \left\{ \mathbf{Q} \mid \lim_{t \rightarrow -\infty} \mathbf{x}_{l_u}(t, \tau; \mathbf{Q}) = (0, 0) \right\} \quad (\text{A.4})$$

is a graph on  $\ell^u(\tau)$  (globally), for any  $\tau \in \mathbb{R}$ . Then the proof follows from a truncation argument.

Using the flow of (S) we get the following.

**Lemma A.2.** *Assume **gu** and **C**. Then the set  $\tilde{W}_{l_u}^u(\tau)$  characterized as in (A.4) is a 1-dimensional immersed submanifold having  $\ell^u(\tau)$  as tangent space in the origin.*

*Proof.* Let us denote by  $\Phi_{T, \tau}$  the diffeomorphism induced by the flow of (S): i.e.  $\Phi_{T, \tau}(\mathbf{Q}) = \mathbf{x}_{l_u}(T, \tau; \mathbf{Q})$ . Then  $\Phi_{T, \tau}(W_{l_u, loc}^u(\tau))$  is a 1-dimensional submanifold for any  $\tau, T \in \mathbb{R}$  and  $\Phi_{T, \tau_1}(W_{l_u, loc}^u(\tau_1)) \supset \Phi_{T, \tau_2}(W_{l_u, loc}^u(\tau_2))$  if  $\tau_1 < \tau_2$ . Hence we may set  $\tilde{W}_{l_u}^u(T) := \bigcup_{\tau \in \mathbb{R}} \Phi_{T, \tau}(W_{l_u, loc}^u(\tau))$  and we see that  $\tilde{W}_{l_u}^u(T)$  is a 1-dimensional immersed manifold, and by construction it is characterized as in (A.4).  $\square$

**Remark A.3.** *We stress that  $\tilde{W}_{l_u}^u(\tau)$  (and  $\tilde{W}_{l_s}^s(\tau)$  constructed below) may be not a usual submanifold in the origin: i.e it may be 8 shaped as in the critical autonomous case, see e.g. Figure 1. However it always contains  $W_{l_u, loc}^u(\tau)$  (respectively  $W_{l_s, loc}^s(\tau)$ ) which is tangent to  $\ell^u(\tau)$  (resp.  $\ell^s(\tau)$ ).*

Now we drop the assumption **C** and we prove Lemma 2.12.

*Proof of Lemma 2.12.* Fix  $\tau \in \mathbb{R}$ ; for every  $\mathbf{Q} \in \mathbb{R}^2$  we can introduce

$$\mathfrak{T}(\mathbf{Q}, \tau) = \sup \{t \mid \mathbf{x}_{l_u}(\cdot, \tau, \mathbf{Q}) \text{ is defined in } [\tau, t]\}.$$

Then  $\lim_{t \rightarrow \mathfrak{T}(\mathbf{Q}, \tau)} |\mathbf{x}_{l_u}(t, \tau, \mathbf{Q})| = +\infty$  if  $\mathfrak{T}(\mathbf{Q}, \tau) < +\infty$ . It is easy to verify that  $\mathfrak{T}(\cdot, \tau)$  is lower semicontinuous, i.e. the sets  $\{\mathbf{Q} \in \mathbb{R}^2 \mid \mathfrak{T}(\mathbf{Q}, \tau) \leq t\}$  are closed. In fact for every  $\mathbf{Q}_0 \in \mathbb{R}^2$  and every  $\varepsilon > 0$  we can find neighbourhoods  $\mathcal{U}$  of  $\mathbf{Q}_0$  and  $\mathcal{V}$  of  $\mathbf{x}_{l_u}(\mathfrak{T}(\mathbf{Q}_0, \tau) - \varepsilon, \tau, \mathbf{Q}_0)$  such that for every  $\mathbf{Q} \in \mathcal{U}$  we have  $\mathbf{x}_{l_u}(\mathfrak{T}(\mathbf{Q}_0, \tau) - \varepsilon, \tau, \mathbf{Q}) \in \mathcal{V}$ , thus giving  $\mathfrak{T}(\mathbf{Q}, \tau) > \mathfrak{T}(\mathbf{Q}_0, \tau) - \varepsilon$  for every  $\mathbf{Q} \in \mathcal{U}$ . Therefore if  $\mathbf{Q}_n \rightarrow \mathbf{Q}_0$ , then  $\liminf_{n \rightarrow \infty} \mathfrak{T}(\mathbf{Q}_n, \tau) \geq \mathfrak{T}(\mathbf{Q}_0, \tau)$ .

Then, we consider

$$\mathfrak{T}^u(\tau) := \inf \{\mathfrak{T}(\mathbf{Q}, \tau) \mid \mathbf{Q} \in W_{l_u, loc}^{u, +}(\tau)\}. \quad (\text{A.5})$$

Notice that  $\mathfrak{T}((0, 0), \tau) = +\infty$  and that  $\mathfrak{T}^u(\tau)$  is increasing by construction. The lower semicontinuity gives us that either one has  $\mathfrak{T}^u(\tau) = +\infty$  or the infimum is in fact a minimum, being  $W_{l_u, loc}^{u, +}(\tau)$  bounded and  $\mathfrak{T}(\mathbf{Q}, \tau) > \tau$ . Moreover the subset containing the points  $\mathbf{Q}$  which *explode to infinity* before a fixed time  $t$ ,

$$\mathcal{X}(t, \tau) = \{\mathbf{Q} \in W_{l_u, loc}^{u, +}(\tau) \mid \mathfrak{T}(\mathbf{Q}, \tau) \leq t\},$$

is a closed subset. Conversely  $\mathcal{W}(t, \tau) = W_{l_u, loc}^{u, +}(\tau) \setminus \mathcal{X}(t, \tau)$  is a relatively open subset of  $W_{l_u, loc}^{u, +}(\tau)$ .

If  $t < \mathfrak{T}^u(\tau)$ , then  $\mathcal{X}(t, \tau) = \emptyset$ , so that  $\tilde{W}_{l_u}^{u, +}(t) := \Phi_{t, \tau} W_{l_u, loc}^{u, +}(\tau)$  is diffeomorph to  $W_{l_u, loc}^{u, +}(\tau)$ , and it is a 1-dimensional manifold with border. In fact it is easy to check that the map  $\Phi_{\tau, T} = \Phi_{T, \tau}^{-1}$  is well defined in an open neighborhood of  $\Phi_{T, \tau} W_{l_u, loc}^{u, +}(\tau)$ .

We assume first for illustrative purpose that, *for any  $\mathbf{Q} = (Q_x, Q_y) \in W_{l_u, loc}^{u, +}(\tau)$  the function  $\mathfrak{T}(\mathbf{Q}, \tau)$  is strictly decreasing in  $Q_x$* : this is the case, e.g., if (S) is autonomous. This assumption will be removed later on.

If we set  $t = \mathfrak{T}^u(\tau)$  we have  $\mathcal{X}(t, \tau) = \{\mathbf{Q}_\tau\}$  where  $\mathbf{Q}_\tau = (\delta, Q_y)$  is the endpoint of  $W_{l_u, loc}^{u, +}(\tau)$ , while if  $t > \mathfrak{T}^u(\tau)$  the sets  $\mathcal{X}(t, \tau)$ , which contains  $\mathbf{Q}_\tau$ , and  $\mathcal{W}(t, \tau)$  are connected. In both the cases, the map  $\Phi_{t, \tau}$  is well defined on  $\mathcal{W}(t, \tau)$ , and the set  $\tilde{W}_{l_u}^{u, +}(t) := \Phi_{t, \tau} \mathcal{W}(t, \tau)$  is diffeomorph to  $\mathcal{W}(t, \tau)$ . In fact there is an open neighborhood of  $\tilde{W}_{l_u}^{u, +}(t)$  which is mapped by the inverse diffeomorphism  $\Phi_{\tau, t}$  into an open neighborhood of  $\mathcal{W}(t, \tau)$ . So  $\tilde{W}_{l_u}^{u, +}(t) \setminus \{(0, 0)\}$  is a 1-dimensional manifold without border, see Figure 3.

Now let us repeat the discussion replacing  $\tau$  by  $\tau_0 < \tau$ . It is easy to check that  $\Phi_{t, \tau_0} \mathcal{W}(t, \tau_0) \supseteq \Phi_{t, \tau} \mathcal{W}(t, \tau)$ , and if  $t \geq \mathfrak{T}^u(\tau)$ , then  $\Phi_{t, \tau_0} \mathcal{W}(t, \tau_0) = \Phi_{t, \tau} \mathcal{W}(t, \tau) = \tilde{W}_{l_u}^{u, +}(t)$ , and it is unbounded. Furthermore by construction

$$\tilde{W}_{l_u}^{u, +}(t) = \{\mathbf{Q} \mid \lim_{t \rightarrow -\infty} \mathbf{x}_{l_u}(t, t, \mathbf{Q}) = (0, 0), \dot{\mathbf{x}}_{l_u}(t, t, \mathbf{Q}) > 0 \text{ for } t \ll 0\}, \quad (\text{A.6})$$

if  $t \geq \mathfrak{T}^u(\tau)$ . Therefore the set

$$W_{l_u}^{u, +}(T) = \bigcup_{\tau \leq T} \Phi_{T, \tau} \mathcal{W}(T, \tau)$$



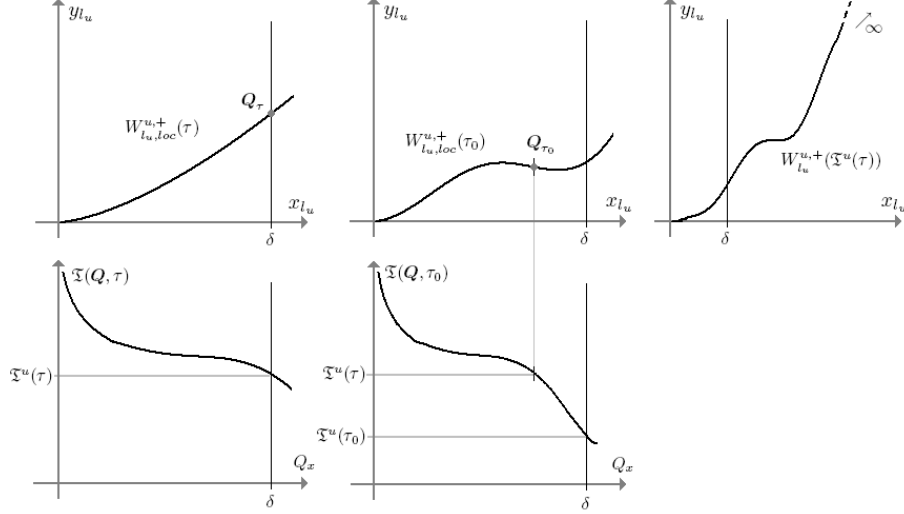


Figure 3: Assume that  $\mathfrak{T}(\mathbf{Q}, \tau)$  is continuous and strictly decreasing in  $Q_x$ . At the time  $\tau$  (on the left), the endpoint  $\mathbf{Q}_\tau = (\delta, Q_y)$  minimizes  $\mathfrak{T}(\cdot, \tau)$  along  $W_{l_u, loc}^{u,+}(\tau)$ , thus having  $\mathcal{W}(\mathfrak{T}^u(\tau), \tau) = W_{l_u, loc}^{u,+}(\tau) \setminus \{\mathbf{Q}_\tau\}$ . If we consider  $\tau_0 < \tau$  (at the center), then  $\mathfrak{T}^u(\tau_0) < \mathfrak{T}^u(\tau)$  and the set  $\mathcal{W}(\mathfrak{T}^u(\tau), \tau_0)$  consists of the points to the left with respect to  $\mathbf{Q}_{\tau_0} = \mathbf{x}_{l_u}(\tau_0, \tau, \mathbf{Q}_\tau)$ . The images  $\Phi_{\mathfrak{T}^u(\tau), \tau} \mathcal{W}(\mathfrak{T}^u(\tau), \tau)$  and  $\Phi_{\mathfrak{T}^u(\tau), \tau_0} \mathcal{W}(\mathfrak{T}^u(\tau), \tau_0)$  gives us the unbounded 1-dimensional manifold  $W_{l_u}^{u,+}(\mathfrak{T}^u(\tau))$  (on the right).

is characterized by the property defined in (A.6).

If we remove the simplifying assumption that  $\mathfrak{T}(\mathbf{Q}, \tau)$  is decreasing with respect to  $Q_x$ , we can repeat the previous discussion with the following changes. The open set  $\mathcal{W}(\mathfrak{T}^u(\tau), \tau)$  can be disconnected, so its image  $\Phi_{\mathfrak{T}^u(\tau), \tau} \mathcal{W}(\mathfrak{T}^u(\tau), \tau)$  may be disconnected too, see Figure 4. However  $\mathcal{W}(\mathfrak{T}^u(\tau), \tau)$  has a connected component containing the origin, say  $\mathcal{W}_1(\mathfrak{T}^u(\tau), \tau)$ , whose image is the connected component of  $\Phi_{\mathfrak{T}^u(\tau), \tau} \mathcal{W}(\mathfrak{T}^u(\tau), \tau)$  containing the origin. Observe that  $\mathcal{W}_1(\mathfrak{T}^u(\tau), \tau)$  is a connected one dimensional manifold, so this property is inherited by  $\Phi_{\mathfrak{T}^u(\tau), \tau} \mathcal{W}(\mathfrak{T}^u(\tau), \tau)$  too.

When  $\mathbf{t} \geq \mathfrak{T}^u(\tau)$ , the map  $\Phi_{\mathbf{t}, \tau}$  is well defined in  $\mathcal{W}(\mathbf{t}, \tau)$  and the set  $\mathcal{X}(\mathbf{t}, \tau)$  may disconnect  $\mathcal{W}(\mathfrak{T}^u(\tau), \tau)$ , see Figure 5. Repeating the previous arguments we see that the image  $\Phi_{\mathbf{t}, \tau} \mathcal{W}(\mathbf{t}, \tau)$  is unbounded and may have many components. However,  $\mathcal{W}(\mathbf{t}, \tau)$  has a connected component containing the origin, say  $\mathcal{W}_1(\mathbf{t}, \tau)$  and we can consider the image  $W_{l_u}^{u,+}(\mathbf{t}) := \Phi_{\mathbf{t}, \tau} \mathcal{W}_1(\mathbf{t}, \tau)$  which is a 1-dimensional connected manifold and it is unbounded.

Again, cf. Figure 4, if we switch from  $\tau$  to  $\tau_0 < \tau$  we see that  $\Phi_{\mathbf{t}, \tau_0} \mathcal{W}_1(\mathbf{t}, \tau_0) \supseteq \Phi_{\mathbf{t}, \tau} \mathcal{W}_1(\mathbf{t}, \tau)$ , and if  $T = \mathfrak{T}^u(\tau)$ , then

$$W_{l_u}^{u,+}(T) = \Phi_{T, \tau} \mathcal{W}_1(T, \tau) = \Phi_{T, \tau_0} \mathcal{W}_1(T, \tau_0) \supseteq \Phi_{T, \tau_1} \mathcal{W}_1(T, \tau_1)$$

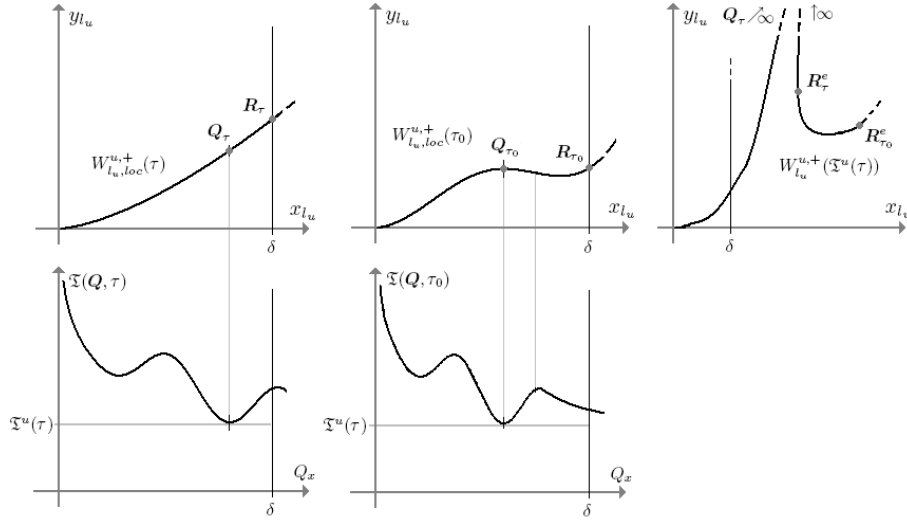


Figure 4: If  $\mathfrak{T}(Q, \tau)$  is not decreasing in  $Q_x$ , at the time  $\tau$  (on the left), the minimum  $\mathfrak{T}^u(\tau)$  can be attained in a point  $Q_\tau = (Q_x^\tau, Q_y^\tau)$  with  $Q_x^\tau < \delta$ , while we denote by  $R_\tau$  the endpoint of  $W_{l_u,loc}^{u,+}(\tau)$ . We consider also  $\tau_0 < \tau$  (in the center), where we can find the point  $Q_{\tau_0} = x_{l_u}(\tau_0, \tau, Q_\tau)$  and we denote by  $R_{\tau_0}$  the endpoint of  $W_{l_u,loc}^{u,+}(\tau_0)$ . In both the situations the sets  $\mathcal{W}(\mathfrak{T}^u(\tau), \tau)$  and  $\mathcal{W}(\mathfrak{T}^u(\tau), \tau_0)$  are disconnected respectively at the point  $Q_\tau$  and  $Q_{\tau_0}$ . The images  $\Phi_{\mathfrak{T}^u(\tau), \tau} \mathcal{W}(\mathfrak{T}^u(\tau), \tau) \subset \Phi_{\mathfrak{T}^u(\tau), \tau_0} \mathcal{W}(\mathfrak{T}^u(\tau), \tau_0)$  gives us two unbounded disconnected sets contained in  $\tilde{W}_{l_u}^{u,+}(\mathfrak{T}^u(\tau))$  (on the right). The second components have endpoints respectively  $R_\tau^e = \Phi_{\mathfrak{T}^u(\tau), \tau}(R_\tau)$  and  $R_{\tau_0}^e = \Phi_{\mathfrak{T}^u(\tau), \tau_0}(R_{\tau_0})$ ; while the point  $Q_\tau$  is, roughly speaking, sent to infinity by the flux  $\Phi_{\mathfrak{T}^u(\tau), \tau}$ .

for any  $\tau_0 < \tau < \tau_1$ . Hence, we can define for every  $T \in \mathbb{R}$  the set

$$W_{l_u}^{u,+}(T) = \bigcup_{\tau \leq T} \Phi_{T, \tau} \mathcal{W}_1(T, \tau)$$

which is a 1-dimensional connected manifold containing the origin in its border, and it is unbounded. Reasoning in the same way we see that if  $\tau_0 < \tau_1$  then

$$\Phi_{T, \tau_0} \mathcal{W}(T, \tau_0) \supseteq \Phi_{T, \tau_1} \mathcal{W}(T, \tau_1),$$

therefore we can define

$$\tilde{W}_{l_u}^{u,+}(T) := \bigcup_{\tau \leq T} \Phi_{T, \tau} \mathcal{W}(T, \tau)$$

Notice that  $\tilde{W}_{l_u}^{u,+}(T)$  may not be a manifold (it may break infinitely many times), since  $\mathcal{W}(T, \tau)$  may be disconnected. However by construction  $\tilde{W}_{l_u}^{u,+}(T)$

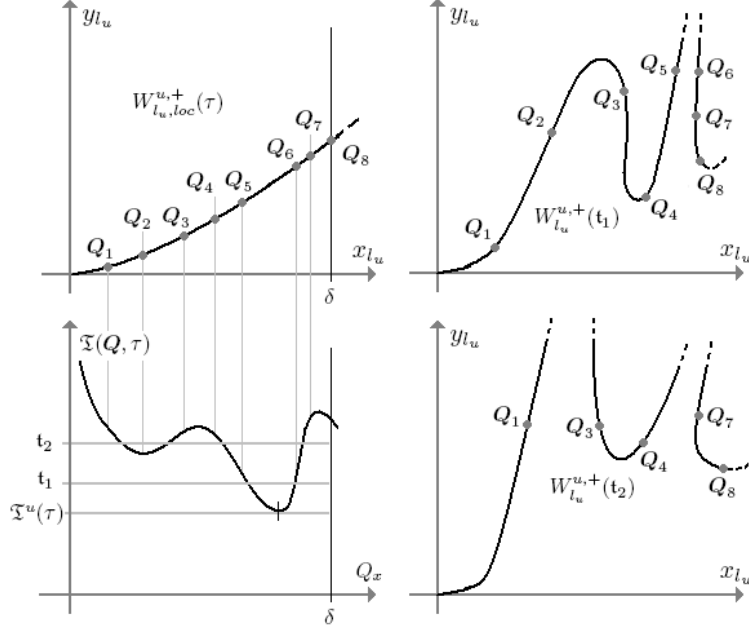


Figure 5: In general  $\mathcal{W}(t, \tau) \subset W_{l_u,loc}^{u,+}(\tau)$  may be disconnected when  $t > \mathfrak{T}^u(\tau)$ . The picture sketches an example. Consider  $t_2 > t_1 > \mathfrak{T}^u(\tau)$ , the set  $\mathcal{W}(t_1, \tau)$  has two connected components, while  $\mathcal{W}(t_2, \tau)$  has three components. On the right we have drawn the corresponding images  $\Phi_{t_1, \tau} \mathcal{W}(t_1, \tau) \subset W_{l_u}^{u,+}(t_1)$  and  $\Phi_{t_2, \tau} \mathcal{W}(t_2, \tau) \subset W_{l_u}^{u,+}(t_2)$ . We show how some points  $Q_1, \dots, Q_8$ , are mapped by the fluxes  $\Phi_{t_1, \tau}$  and  $\Phi_{t_2, \tau}$ , denoting the images again with  $Q_1, \dots, Q_8$  for simplicity. In particular, at the time  $t_2$  the solution  $x_{l_u}(\cdot, \tau, Q_j)$  is not defined for  $j = 2, 5, 6$ .

may be still characterized as in (A.6); moreover  $W_{l_u}^{u,+}(T)$  is the connected component of  $\tilde{W}_{l_u}^{u,+}(T)$  containing the origin in its border, and it is, as shown above, a 1-dimensional connected manifold.

The construction of  $W_{l_u}^{u,-}(\tau)$  and of  $W_{l_u}^u(\tau) = W_{l_u}^{u,-}(\tau) \cup W_{l_u}^{u,+}(\tau)$  is completely analogous and it is omitted. This concludes the part of the proof of Lemma 2.12 concerning the unstable manifolds. The construction of the stable leaves is very similar and we just sketch it.

With a specular argument we assume **gs**, so that  $\mathcal{A}_{l_s}(+\infty)$  has  $\nu_2 < 0 < \nu_1$  as eigenvalues, where  $\nu_1 := \alpha_{l_u} - \kappa(\beta)$  and  $\nu_2 := \alpha_{l_s} + 2 - n + \kappa(\beta)$ . So, let  $Y(t)$  be the fundamental matrix of (A.1), where  $\mathcal{A}_{l_u}(t)$  is replaced by  $\mathcal{A}_{l_s}(t)$ . Then, for any  $\tau \in \mathbb{R}$  there is a constant  $K = K(\tau) > 1$ , and a projection  $\mathcal{P}^+$  such that

$$\begin{aligned} \|Y(t)(I - \mathcal{P}^+)Y(s)^{-1}\| &\leq Ke^{\nu_1(t-s)} & \text{for any } s > t > \tau, \\ \|Y(t)\mathcal{P}^+Y(s)^{-1}\| &\leq Ke^{\nu_2(t-s)} & \text{for any } t > s > \tau, \end{aligned} \quad (\text{A.7})$$

see again [12, Section 4], and [8, Appendix]. Denote by  $\mathcal{P}^+(\tau) := Y(\tau)\mathcal{P}^+Y(\tau)^{-1}$ ,

and by  $\ell^s(\tau)$  the 1-dimensional range of  $\mathcal{P}^+(\tau)$ . Then the solution  $\vec{\xi}(t)$  of (A.1), with  $l_s$  replacing  $l_u$ , is bounded for  $t \geq 0$  iff  $\vec{\xi}(0) \in \ell^s(\tau)$ . Moreover  $\|\vec{\xi}(t)\|e^{-\nu_2 t} \rightarrow c$  as  $t \rightarrow +\infty$  for a suitable  $c > 0$ . This way we are able to construct a local manifold  $W_{l_s,loc}^s(\tau)$  and to reprove a result analogous to Lemma A.1. Then, assuming temporarily **C** and reasoning as in Lemma A.2, we see that  $\Phi_{T,\tau}(W_{l_s,loc}^s(\tau))$  is a 1-dimensional submanifold for any  $\tau, T \in \mathbb{R}$ ; moreover  $\Phi_{T,\tau_2}(W_{l_s,loc}^s(\tau_2)) \supset \Phi_{T,\tau_1}(W_{l_s,loc}^s(\tau_1))$  if  $\tau_1 < \tau_2$ . Hence, assuming **C** and **gs**, we obtain that the set

$$\tilde{W}_{l_s}^s(\tau) := \bigcup_{\tau_0 \geq \tau} \Phi_{\tau,\tau_0}(W_{l_s,loc}^s(\tau_0)) = \{\mathbf{Q} \mid \lim_{t \rightarrow +\infty} \mathbf{x}_{l_s}(t, \tau, \mathbf{Q}) = (0, 0)\} \quad (\text{A.8})$$

is a 1-dimensional immersed manifold having  $\ell^s(\tau)$  as tangent space in the origin.

Then we remove assumption **C** and, arguing as above, we see that  $\tilde{W}_{l_s}^s(\tau)$  may be disconnected, but its connected component containing the origin, denoted by  $W_{l_s}^s(\tau)$ , is again a 1-dimensional manifold. Then repeating the previous discussion we conclude the proof of Lemma 2.12. The part of the proof concerning Lemmas 2.5 and 2.13 is given below.  $\square$

Now we proceed with the proof of Lemma 2.13, which includes Lemma 2.5 as a particular case. The proof is adapted from [13, Lemma 2.10] where it is developed assuming **C** and  $h(r) \equiv 0$ .

*Proof of Lemma 2.13*

Assume **gu** and **gs**; recalling that  $\mathbf{x}_{l_u}(t) = (u(e^t)e^{\alpha_{l_u}t}, u'(e^t)e^{(1+\alpha_{l_u})t})$ , we find  $\mathbf{x}_{l_s}(t) = \mathbf{x}_{l_u}(t)e^{(\alpha_{l_s}-\alpha_{l_u})t}$ . Therefore in particular  $\mathbf{R} = \mathbf{Q}\exp[-(\alpha_{l_u} - \alpha_{l_s})\tau]$ .

Assume first **C** for simplicity. From roughness of exponential dichotomy, cf [12, Chapter 4] and [27, Theorem 2.16], we see that, if  $\mathbf{Q} \in W_{l_u}^u(\tau)$ , then there is  $d = d(\mathbf{Q})$  such that  $\lim_{t \rightarrow -\infty} \mathbf{x}_{l_u}(t, \tau, \mathbf{Q})e^{-(\alpha_{l_u}-\kappa(\eta))t} = d(1, -\kappa(\eta))$ . Assume  $d > 0$  for definiteness; then for the corresponding solution  $u(r)$  of (Hr) we get

$$u(r) = x_{l_u}(\ln(r), \tau, \mathbf{Q})r^{-\alpha_{l_u}+\kappa(\eta)} \rightarrow d, \quad \text{as } r \rightarrow 0. \quad (\text{A.9})$$

Assume now  $\mathbf{Q} \notin W_{l_u}^u(\tau)$ . Then, if  $l_u \neq 2^*$ , we find that  $|\mathbf{x}_{l_u}(t, \tau, \mathbf{Q})|$  is uniformly positive as  $t \rightarrow -\infty$ , and if  $l_u = 2^*$  there is a sequence  $t_n \rightarrow -\infty$  such that  $|\mathbf{x}_{l_u}(t_n, \tau, \mathbf{Q})|$  is uniformly positive: in both the cases the corresponding solution  $u(r)$  of (Hr) is not a  $\mathcal{R}$ -solution since  $u(r)r^{\alpha_{l_u}-\kappa(\eta)} \not\rightarrow 0$  as  $r \rightarrow 0$ . Further we easily see that  $\mathbf{Q} \in W_{l_u}^u(\tau) \iff \mathbf{R} \in W_{l_s}^u(\tau)$ .

Arguing similarly, if  $\mathbf{Q} \in W_{l_s}^s(\tau)$  then there exists  $L = L(\mathbf{Q}) > 0$  such that  $\lim_{t \rightarrow +\infty} \mathbf{x}_{l_s}(t, \tau, \mathbf{Q})e^{-[\gamma_{l_s}+\kappa(\beta)]t} = L(1, -(n-2)+\kappa(\beta))$ : hence the corresponding solution  $u(r)$  of (Lr) satisfies

$$\lim_{t \rightarrow +\infty} u(e^t)e^{(\alpha_{l_s}-\gamma_{l_s}-\kappa(\beta))t} = \lim_{r \rightarrow +\infty} u(r)r^{n-2-\kappa(\beta)} = L. \quad (\text{A.10})$$

So we can easily conclude as above.

Hence, if we assume either **Gu** or **gu**, we can construct the unstable manifold  $W_{l_u}^u(\tau)$  for any  $\tau \in \mathbb{R}$ ; similarly if either **Gs** or **gs** hold, we can construct the

stable manifold  $W_{l_s}^s(\tau)$  for any  $\tau \in \mathbb{R}$ . Moreover Remark 2.2 still holds and we can construct  $W_{l_u}^s(\tau)$  and  $W_{l_s}^u(\tau)$  via (2.7) too.

Now we drop assumption **C**. In this case, due to the presence of non-continuable trajectories, we need to distinguish between  $W_{l_u}^u(\tau)$  and  $\tilde{W}_{l_u}^u(\tau)$ , and similarly for the other manifolds. In fact  $\mathbf{x}_{l_u}(t, \tau, \mathbf{Q}) \rightarrow (0, 0)$  iff  $\mathbf{Q} \in \tilde{W}_{l_u}^u(\tau)$ . Further for any  $\mathbf{Q} \in \tilde{W}_{l_u}^u(\tau)$  we can find  $N \gg 1$  such that  $\mathbf{x}_{l_u}(T, \tau, \mathbf{Q}) \in W_{l_u}^u(T)$  for any  $T \leq -N$ . So we can repeat the previous argument and we see that the corresponding solution  $u(r)$  of (Hr) is a  $\mathcal{R}$ -solution. A similar argument holds for the stable manifold.

So for any  $\tau$  we find that  $\mathbf{Q} \in W_{l_u}^u(\tau)$  iff  $\mathbf{R} \in W_{l_s}^u(\tau)$  iff  $u(r) = u(r, d)$  is a  $\mathcal{R}$ -solution with  $0 < d < d_\tau^+$ , see (2.9). Similarly  $\mathbf{Q} \in W_{l_u}^s(\tau)$  iff  $\mathbf{R} \in W_{l_s}^s(\tau)$  if  $u(r) = u(r, L)$  is a  $\mathcal{R}$ -solution with  $0 < L < L_\tau^+$ .  $\square$

## A.2 Proof of Lemmas 3.3 and 3.6

We prove now Lemma 3.3: such a result has been obtained in presence of continuity of the solutions and for  $h(r) \equiv 0$  in [13, Lemma 2.10].

*Proof of Lemma 3.3.* We will prove only the first part of the statement, the second follows similarly. Consider the parametrization  $\Sigma_{l_u}^{u,+}(\cdot, T)$ . Assume first **C**. Observe that, starting from  $\Sigma_{l_u}^{u,+}(\cdot, T)$ , we can construct a parametrization of  $W_{l_u}^u(\tau)$  for any  $\tau \in \mathbb{R}$ , by setting  $\Sigma_{l_u}^{u,+}(\omega, \tau) := \mathbf{x}_{l_u}(\tau; T, \Sigma_{l_u}^{u,+}(\omega, T))$ . In fact, the function  $\Sigma_{l_u}^{u,+} : [0, +\infty) \times \mathbb{R} \rightarrow \mathbb{R}^2$  is continuous in both the variables, and the map  $(\omega, \tau) \mapsto (\Sigma_{l_u}^{u,+}(\omega, \tau), z(\tau))$  is injective in  $\mathbf{W}^u$ . According to this parametrization,  $\mathbf{x}_{l_u}(t; \tau, \Sigma_{l_u}^{u,+}(\omega, \tau))$  coincides with  $\mathbf{x}_{l_u}(t; T, \Sigma_{l_u}^{u,+}(\omega, T))$  and corresponds to the given solution  $u(r, d(\omega))$  for any  $\tau \in \mathbb{R}$ .

Fix  $N \in \mathbb{R}$  and let  $\delta := \delta(N)$  be the constant defined in Lemma A.1; we can find  $\bar{\omega} > 0$  and  $N(\bar{\omega}) < N$  such that  $\Sigma_{l_u}^{u,+}(\omega, \tau) \in W_{l_u, \text{loc}}^{u,+}(\tau)$ , whenever  $0 \leq \omega \leq \bar{\omega}$  and  $\tau \leq N(\bar{\omega})$ .

We now show that  $d(\omega)$  is strictly increasing. Once proved this claim for this particular parametrization we have it for any parametrization  $\varpi \rightarrow \Sigma_{l_u}^{u,+}(\varpi, \tau)$  of  $W_{l_u}^{u,+}(\tau)$  as in the assumption of Lemma 3.3, due to the monotonicity of the change of variables  $\varpi(\omega)$ . Using Lemma A.1 we see that we can choose  $\omega_1 < \omega_2$ , so that  $\Sigma_{l_u}^{u,+}(\omega_i, \tau) \in W_{l_u, \text{loc}}^{u,+}(\tau)$  for any  $\tau \leq N(\omega_2)$  and for  $i = 1, 2$ . Hence  $\Sigma_{l_u}^{u,+}([0, \omega_2] \times \{\tau\}) \subset W_{l_u}^{u,+}(\tau)$  is a graph on  $\ell^u(\tau)$ , for any  $\tau \leq N(\omega_2)$ , see Lemma A.1. In particular  $x_{l_u}(t; T, \Sigma_{l_u}^{u,+}(\omega_1, T)) - x_{l_u}(t; T, \Sigma_{l_u}^{u,+}(\omega_2, T)) < 0$  for  $t = N(\omega_2)$ . We claim that  $W_{l_u, \text{loc}}^{u,+}(\tilde{\tau})$  is a graph on a segment of the  $x$  axis, for any  $\tilde{\tau} \leq N(\omega_2)$ . In fact when  $\eta = 0$  the claim is obvious since  $\ell^u(\tau)$  is contained in the  $x$  axis. If  $\eta \neq 0$ , since  $\ell^u(\tau)$  is not orthogonal to the  $x$  axis, possibly choosing a smaller  $\delta$  we can again assume that  $W_{l_u, \text{loc}}^{u,+}(\tilde{\tau})$  is a graph on the  $x$  axis too, so the claim is true.

Assume for contradiction that  $d(\omega_1) > d(\omega_2)$ , then there is  $\tilde{\tau} < N(\omega_2)$  such that  $x_{l_u}(t; T, \Sigma_{l_u}^{u,+}(\omega_1, T)) - x_{l_u}(t; T, \Sigma_{l_u}^{u,+}(\omega_2, T))$  is positive for any  $t < \tilde{\tau}$  and it is zero for  $t = \tilde{\tau}$ . In particular  $W_{l_u, \text{loc}}^{u,+}(\tilde{\tau})$  is not a graph on the  $x$  axis,

so we have found a contradiction. Hence  $d(\omega_1) < d(\omega_2)$ , and the Lemma is concluded if **C** holds. Notice that we can redefine the parametrization and use directly  $d$  instead of  $\omega$  as parameter, so, with a little abuse of notation we find the parametrization  $\Sigma_{l_u}^{u,+}(d, \tau)$  of  $W_{l_u}^{u,+}(\tau)$  which is continuous (and smooth) in both the variables for any  $(d, \tau) \in [0, +\infty) \times \mathbb{R}$ .

Now we drop **C**. Fix  $T \in \mathbb{R}$ , and correspondingly  $d_T^+$  as in (2.9), so that, for any  $d \in (0, d_T^+)$ ,  $u(r, d)$  is continuable for any  $0 < r < e^T$ . Using the previous discussion and a truncation argument, for any  $D \in (0, d_T^+)$ , we can define the map  $\Sigma_{l_u}^{u,+}(d, T)$  for  $d \in [0, D]$ ; so we get a parametrization of a connected branch of  $W_{l_u}^{u,+}(T)$ , say  $\bar{W}_{l_u}^{u,+}(T)$ . Since for any point  $\mathbf{Q} \in \bar{W}_{l_u}^{u,+}(T)$  we have that  $\mathbf{x}_{l_u}(\tau; T, \mathbf{Q})$  exists for any  $\tau \leq T$ , arguing as above, we find that  $\Sigma_{l_u}^{u,+}(d, \tau) = \mathbf{x}_{l_u}(\tau, T, \Sigma_{l_u}^{u,+}(d, T))$  is a parametrization of a connected branch of  $W_{l_u}^{u,+}(\tau)$ , denoted again by  $\bar{W}_{l_u}^{u,+}(\tau)$ , for any  $\tau \leq T$ .

Now let  $\tau > T$  and notice that the function  $d^+(\tau) = d_\tau^+$  defined in (2.9) is decreasing in  $\tau$ . If  $D < d_\tau^+$ , reasoning as above, we find that  $\Sigma_{l_u}^{u,+}(d, \tau) = \mathbf{x}_{l_u}(\tau, T, \Sigma_{l_u}^{u,+}(d, T))$  gives again a parametrization of a connected branch of  $W_{l_u}^{u,+}(\tau)$ , for  $0 \leq d \leq D$ . If  $D \geq d_\tau^+$  then  $\Sigma_{l_u}^{u,+}(d, \tau) = \mathbf{x}_{l_u}(\tau, T, \Sigma_{l_u}^{u,+}(d, T))$  for  $0 < d < d_\tau^+$  is unbounded and it is itself a parametrization of the whole manifold  $W_{l_u}^{u,+}(\tau)$ . Assume  $D < d_\tau^+$ , this way we obtain a map  $\Sigma_{l_u}^{u,+}(d, \tau)$  which is continuous in both the variables for

$$\bar{E} = ([0, D] \times (-\infty, T] \cup \{(d, \tau) \mid 0 < d < \max\{D, d_T^+\}, \tau \geq T\}).$$

Further  $d \rightarrow \Sigma_{l_u}^{u,+}(d, \tau)$  is injective for  $(d, \tau) \in \bar{E}$ . For the arbitrariness of  $D < d_T^+$  we can let  $D \rightarrow d_T^+$ . Then from the arbitrariness of  $T \in \mathbb{R}$  we define  $\Sigma_{l_u}^{u,+}$  in the whole  $E = \{(d, \tau) \mid 0 < d < d_\tau^+, \tau \in \mathbb{R}\}$ , it is continuous in both the variables and it gives a bijective parametrization of  $W_{l_u}^{u,+}(\tau)$  for any  $\tau \in \mathbb{R}$ .  $\square$

Now we prove Lemma 3.6: the argument is a modification of [13, Propositions 3.5, 3.8]. In fact, in this setting, we need to take into account the fact that  $\ell^u(\tau)$  and  $\ell^s(\tau)$  change with  $\tau$  (due to the presence of Hardy potentials), while in [13] there was not this difficulty. In particular we need to ask for  $l_s > 2^*$  and to profit of Lemma 3.2.

*Proof of Lemma 3.6.* We introduce some definitions borrowed from [3, 24]. Following [3, 24, 30], given a curve  $\gamma : [a, b] \rightarrow \mathbb{R}^2 \setminus \{(0, 0)\}$ , we define its rotation number  $w(\gamma)$  by setting

$$w(\gamma) := \text{Int} \left[ \frac{\theta_\gamma(b) - \theta_\gamma(a)}{2\pi} \right], \quad (\text{A.11})$$

where  $\text{Int}[\cdot]$  denotes the integer part, and  $\gamma(t) = (\rho_\gamma(t) \cos \theta_\gamma(t), \rho_\gamma(t) \sin \theta_\gamma(t))$ . As pointed out in [24], we can extend this definition to a curve  $\gamma$  defined in a semi-open interval  $[a, b)$  if  $\lim_{t \rightarrow b-} \theta_\gamma(t)$  exists (even if it is infinite).

Our argument will be rather sketchy since we just adapt [13, 24]. Let  $\gamma^i(t) : [a, b] \rightarrow \mathbb{R}^2$ , for  $i = 1, 2$ , be curves in  $\mathbb{R}^2$  which do not intersect each other, and

let  $\phi(t)$  be a smooth monotone function such as  $\varphi(t) = z(t) = e^{\varpi t}$  as in (2.4), or  $\varphi(t) = \zeta(t) = e^{-\varpi t}$  as in (2.5) or  $\varphi(t) = t$  as in [3]. Then  $\mathbf{\Gamma}^i(t) = (\gamma^i(t), \varphi(t))$  are curves in  $\mathbb{R}^3$ . Following [3], we call linking number of  $\gamma^1, \gamma^2$  in  $[a, b]$  the number  $w(\gamma^1 - \gamma^2)$ , i.e. the number of complete rotations of a curve around the other. Such a quantity is invariant for homotopies in  $\mathbb{R}^3$  which preserve the endpoints  $\mathbf{\Gamma}^1(a) = \mathbf{\Gamma}^2(a)$  and  $\mathbf{\Gamma}^1(b) = \mathbf{\Gamma}^2(b)$ .

We want to establish an homotopy between two curves so that linking number and rotation number are equal. Let us fix  $\tau \in \mathbb{R}$  and  $\mathbf{Q} \in W_{l_s}^{s,+}(\tau)$ . Since  $\mathbf{x}_{l_s}(t, \tau, \mathbf{Q})$  converges to the origin as  $t \rightarrow +\infty$  and  $\dot{x}_{l_s}(t, \tau, \mathbf{Q}) < 0$  for  $t \gg 1$ , for every  $\delta > 0$  we can find  $a = a(\delta) \gg 1$  such that  $|\mathbf{x}_{l_s}(a, \tau, \mathbf{Q})| = \delta$  and  $|\mathbf{x}_{l_s}(t, \tau, \mathbf{Q})| > \delta$  for  $t \in [\tau, a]$ .

Then we set  $\gamma^1(t) = \mathbf{x}_{l_s}(t, \tau, \mathbf{Q})$ , where  $\mathbf{Q} \in W_{l_s}^{s,+}(\tau)$ , and we consider the trajectory  $\mathbf{\Gamma}^1(t) = (\mathbf{x}_{l_s}(t, \tau, \mathbf{Q}), \zeta(t))$  of (2.4) for  $t \in [\tau, a]$ .

We shrink further  $\delta \leq \delta(\tau)$  so that the sets  $W_{l_s,loc}^s(T)$  defined in Lemma A.1 are graphs on  $\ell^s(T)$  for any  $T \geq \tau$ , and we denote by  $\bar{\mathbf{C}}(T)$  the unique point in  $W_{l_s,loc}^{s,+}(T) \cap \{x = \delta\}$ . Let  $0 < L_a(T) < L_b$  be such that  $\Upsilon_{l_s}^{s,+}(L_a(T), T) = \bar{\mathbf{C}}(T)$  and  $\Upsilon_{l_s}^{s,+}(L_b, T) = \mathbf{x}_{l_s}(T, \tau, \mathbf{Q})$ . We consider the curves  $\Psi_1(T) = (\bar{\mathbf{C}}(T), \zeta(T))$  for  $T \in [\tau, a]$ , the curve  $\Psi_2(d) = (\Upsilon_{l_s}^{s,+}(d, \tau), \zeta(\tau))$  for  $d \in [L_a(T), L_b]$  and the curve  $\mathbf{\Gamma}^2(t)$  obtained following the graph of  $\Psi_1$  and then the graph of  $\Psi_2$ . An homotopy between  $\mathbf{\Gamma}^1$  and  $\mathbf{\Gamma}^2$  is obtained by projecting  $\mathbf{\Gamma}^1$  on  $W_{l_s}^s(\tau)$  following the 2-dimensional manifold  $\mathbf{W}^s$ : we sketch the construction, see [24, Lemma 4.3] for more details.

For any  $T \in [\tau, a]$  we construct the function  $H0(\cdot, T)$  obtained following  $\Psi_1(S) = (\bar{\mathbf{C}}(S), \zeta(S))$  for  $S \in [T, a]$ , then  $(\Upsilon_{l_s}^{s,+}(d, T), \zeta(T))$  for  $d \in [L_a(T), L_b]$  and finally  $\mathbf{\Gamma}^1(t) = (\mathbf{x}_{l_s}(t, \tau, \mathbf{Q}), \zeta(t))$  for  $t \in [\tau, T]$  i.e.

$$H0(S, T) = \begin{cases} \Psi_1(a + S(T - a)/L_a(T)) & \text{if } S \in [a, L_a(T)] \\ (\Upsilon_{l_s}^{s,+}(S, T), \zeta(T)) & \text{if } S \in [L_a(T), L_b] \\ (\mathbf{x}_{l_s}(S + (T - L_b)), \tau, \mathbf{Q}), \zeta(T) & \text{if } S \in [L_b, \tau - (T - L_b)] \end{cases}$$

Note that all the curves  $S \mapsto H0(S, T)$  have the same endpoints and are homotopic; moreover  $H0(\cdot, a)$  is  $\mathbf{\Gamma}^1$  and  $H0(\cdot, \tau)$  is  $\mathbf{\Gamma}^2$ . So the linking number of  $\mathbf{\Gamma}^1$  and  $\mathbf{\Gamma}^2$  is 0.

Let us denote by  $\theta_{\Upsilon}(\tau, a)$ ,  $\theta_{\bar{\mathbf{C}}}(\tau, a)$ ,  $\theta_{\mathbf{x}}(\tau, a)$  respectively the angles performed by  $\Upsilon_{l_s}^{s,+}(L, \tau)$  for  $L \in [L_a(T), L_b]$ , by  $\bar{\mathbf{C}}(T)$  for  $T \in [\tau, a]$ , and by  $\mathbf{x}_{l_s}(t, \tau, \mathbf{Q})$  for  $t \in [\tau, a]$ . We have shown that

$$\theta_{\Upsilon}(\tau, a) + \theta_{\bar{\mathbf{C}}}(\tau, a) = \theta_{\mathbf{x}}(\tau, a). \quad (\text{A.12})$$

So Lemma 3.6 follows from (A.12), being (3.9) equivalent.  $\square$

### A.3 On the Wazewski's principle

We conclude the appendix with a result, inspired by Wazewski's principle, which allows to locate the unstable and the stable manifolds. Consider

$$\dot{x} = F(x, t), \quad (\text{A.13})$$

where  $x \in \mathbb{R}^2$ ,  $t \in \mathbb{R}$ ,  $F$  continuous, and assume that the origin  $\mathbf{O} = (0, 0)$  is a critical point for (A.13).

Let  $\mathcal{T}(\tau)$  be a closed set diffeomorphic to a full triangle. We call the vertices  $\mathbf{O}$ ,  $\mathbf{A}(\tau)$  and  $\mathbf{B}(\tau)$ , and  $o(\tau)$ ,  $a(\tau)$ ,  $b(\tau)$  the edges (without endpoints) which are opposite to the respective vertex. Let  $\hat{\mathcal{T}}(\tau)$  denote a further set diffeomorphic to a full triangle having  $\mathbf{O}$  as vertex and with edges  $\hat{a}(\tau) \supset a(\tau)$  and  $\hat{b}(\tau) \supset b(\tau)$ ; it follows that  $\hat{\mathcal{T}}(\tau) \supset \mathcal{T}(\tau)$ . We begin from a result requiring very weak regularity properties.

**Lemma A.4.** *Assume that local uniqueness for the solutions of (A.13) is ensured for any trajectory starting from  $\mathcal{T}(\tau) \setminus \{\mathbf{O}\}$ .*

*Suppose that the flow on  $a(\tau) \cup b(\tau)$  points towards the interior of  $\mathcal{T}(\tau)$ , and on  $o(\tau)$  points towards the exterior of  $\mathcal{T}(\tau)$  for any  $t \leq \tau$ . Assume further that the flow on  $\{\mathbf{A}(\tau), \mathbf{B}(\tau)\}$  points towards the interior of  $\hat{\mathcal{T}}(\tau)$  for any  $t \leq \tau$ . Finally suppose that if a solution  $\mathbf{x}(t)$  of (A.13) satisfies  $\mathbf{x}(t) \in \mathcal{T}(\tau)$  for any  $t \leq \tau$ , then  $\lim_{t \rightarrow -\infty} \mathbf{x}(t) = \mathbf{O}$ .*

*Then there is a compact connected set  $\bar{W}^u(\tau) \subset \mathcal{T}(\tau)$  such that  $\mathbf{O} \in \bar{W}^u(\tau)$ ,  $\bar{W}^u(\tau) \cap o(\tau) \neq \emptyset$ , with the following property:*

$$\bar{W}^u(\tau) \subset \{\mathbf{Q} \mid \lim_{t \rightarrow -\infty} \mathbf{x}(t, \tau; \mathbf{Q}) = \mathbf{O}, \mathbf{x}(t, \tau; \mathbf{Q}) \in \mathcal{T}(\tau) \text{ for any } t \leq \tau\}.$$

*Proof.* This Lemma is proved in [21, § 3], see also [22, Lemma 3.5]: the reasoning relies on a connection argument and a topological idea developed in [36, Lemma 4].  $\square$

Obviously the same idea can be applied to construct stable sets.

**Lemma A.5.** *Assume that local uniqueness for the solutions of (A.13) is ensured for any trajectory starting from  $\mathcal{T}(\tau) \setminus \{\mathbf{O}\}$ .*

*Suppose that the flow on  $a(\tau) \cup b(\tau)$  points towards the exterior of  $\mathcal{T}(\tau)$ , and on  $o(\tau)$  points towards the interior of  $\mathcal{T}(\tau)$  for any  $t \geq \tau$ . Assume further that the flow on  $\{\mathbf{A}(\tau), \mathbf{B}(\tau)\}$  points towards the exterior of  $\hat{\mathcal{T}}(\tau)$  for any  $t \geq \tau$ . Finally suppose that if a solution  $\mathbf{x}(t)$  of (A.13) satisfies  $\mathbf{x}(t) \in \mathcal{T}(\tau)$  for any  $t \geq \tau$ , then  $\lim_{t \rightarrow +\infty} \mathbf{x}(t) = \mathbf{O}$ .*

*Then there is a compact connected set  $\bar{W}^s(\tau) \subset \mathcal{T}(\tau)$  such that  $\mathbf{O} \in \bar{W}^s(\tau)$ ,  $\bar{W}^s(\tau) \cap o(\tau) \neq \emptyset$ , with the following property:*

$$\bar{W}^s(\tau) \subset \{\mathbf{Q} \mid \lim_{t \rightarrow +\infty} \mathbf{x}(t, \tau; \mathbf{Q}) = \mathbf{O}, \mathbf{x}(t, \tau; \mathbf{Q}) \in \mathcal{T}(\tau) \text{ for any } t \geq \tau\}.$$

If we are in the position to apply invariant manifold theory for non-autonomous systems, clearly we find that these sets are manifolds. So we get the following.

**Lemma A.6.** *Assume that we are in the hypotheses of Lemma A.4, respectively of Lemma A.5. Assume further that  $F$  is  $C^1$  and it is continuous in  $x$  uniformly with respect to  $t \in \mathbb{R}$ . Suppose that the linearized system admits exponential dichotomy, i.e. there are projections  $\mathcal{P}^+$  and  $\mathcal{P}^-$  of rank 1 such that (A.2)*



and (A.7) hold, so that  $\mathcal{O}$  admits unstable and stable manifolds  $W^u(\tau)$  and  $W^s(\tau)$  for any  $\tau \in \mathbb{R}$ . Then the set  $\bar{W}^u(\tau) \subset (\mathcal{T}(\tau) \cap W^u(\tau))$  constructed in Lemma A.4, resp. the set  $\bar{W}^s(\tau) \subset (\mathcal{T}(\tau) \cap W^s(\tau))$  constructed in Lemma A.5, is a connected 1-dimensional manifold.

## References

- [1] S. Bae, On positive entire solutions of indefinite semilinear elliptic equations, J. Differential Equations 247, no. 5, 1616-1635 (2009).
- [2] S. Bae, Classification of positive solutions of semilinear elliptic equations with Hardy term, Discrete Contin. Dyn. Syst. Supplement, 31-39 (2013).
- [3] R. Bamon, M. Del Pino and I. Flores, Ground states of semilinear elliptic equations: a geometric approach, Ann. Inst. Henry Poincaré 17, 551-581 (2000).
- [4] F. Battelli and R. Johnson, On positive solutions of the scalar curvature equation when the curvature has variable sign, Nonlinear Anal. 47, 1029-1037 (2001).
- [5] G. Bianchi, Non-existence of positive solutions to semilinear elliptic equations on  $\mathbb{R}^n$  or  $\mathbb{R}_+^n$  through the method of moving planes, Comm. Partial Differential Equations 22, 1671-1690 (1997).
- [6] D. Bonheure, J. Gomes, P. Habets, Multiple positive solutions of superlinear elliptic problems with sign-changing weight J. Differential Equations 214, no. 1, 36-64, (2005),.
- [7] P. Cac, A. Fink and J. Gatica, Nonnegative solutions of the radial laplacian with nonlinearity that changes sign Proc. Amer. Math. Soc. 123, 1393-1398 (1995).
- [8] A. Calamai and M. Franca, Mel'nikov methods and homoclinic orbits in discontinuous systems, J. Dyn. Differential Equations 25, no. 3, 733-764 (2013).
- [9] F. C. Cîrstea: A complete classification of the isolated singularities for nonlinear elliptic equations with inverse square potentials, Mem. Amer. Math. Soc. 227, no. 1068, (2014).
- [10] E. Coddington and N. Levinson: Theory of Ordinary Differential Equations, Mc Graw Hill, New York, (1955).
- [11] W.A. Coppel: Stability and asymptotic behavior of differential equations, Heath Mathematical Monograph, (1965).
- [12] W.A. Coppel: Dichotomies in stability theory, Lecture Notes in Mathematics 377, Springer, Berlin, (1978).

- [13] F. Dalbono and M. Franca, Nodal solutions for supercritical Laplace equations, Comm. in Math. Phys. <http://dx.doi.org/10.1007/s00220-015-2546-y>
- [14] Y.-B. Deng, Y. Li, Y. Liu, On the stability of the positive radial steady states for a semilinear Cauchy problem, Nonlinear Anal. 54, 291-318 (2003).
- [15] Z. Dosla, M. Marini and S. Matucci, *Positive solutions of nonlocal continuous second order BVP's*, Dynam. Systems Appl. 23, no. 2-3, 431-446 (2014).
- [16] V. Felli, E. Marchini, S. Terracini, On the behavior of solutions to Schrödinger equations with dipole type potentials near the singularity, Discrete Contin. Dyn. Syst. 21, no. 1, 91-119 (2008).
- [17] V. Felli, E. Marchini and S. Terracini, Fountain-like solutions for non-linear elliptic equations with critical growth and Hardy potential, Commun. Contemp. Math. 7, no. 6, 867-904 (2005).
- [18] A. Ferrero and F. Gazzola, Existence of solutions for singular critical growth semilinear elliptic equations, J. Differential Equations 177, 494-522 (2001).
- [19] R.H. Fowler, Further studies of Emden's and similar differential equations, Quart. J. Math. 2, 259-288 (1931).
- [20] M. Franca, Some results on the  $m$ -Laplace equations with two growth terms; J. Dynamics Differential Equations, 17, 391-425 (2005). <http://dx.doi.org/10.1007/s10884-005-4572-5>
- [21] M. Franca, Non-Autonomous Quasilinear Elliptic Equations and Wazewski's principle, Topol. Methods Nonlinear Anal. 23, 213-235 (2004).
- [22] M. Franca, Radial ground states and singular ground states for a spatial dependent  $p$ -Laplace equation, J. Differential Equations 248, 2629-2656 (2010).
- [23] M. Franca, Positive solutions for semilinear elliptic equations with mixed non-linearities: 2 simple models exhibiting several bifurcations, J. Dynam. Differential Equations 23, 573-611 (2011).
- [24] M. Franca, Positive solutions of semilinear elliptic equations: a dynamical approach, Differential Integral Equations 26, 505-554 (2013).
- [25] M. Franca and A. Sfecci, On a diffusion model with absorption and production, [arXiv:1606.02196](https://arxiv.org/abs/1606.02196)

- [26] B. Gidas, W.M. Ni and L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in  $\mathbb{R}^n$ , Adv. Math. Suppl. Studies 7A, 369-402 (1981).
- [27] R. Johnson, *Concerning a theorem of Sell*, J. Differential Equations 30, 324-339 (1978).
- [28] R. Johnson, X.B. Pan and Y.F. Yi, Singular ground states of semilinear elliptic equations via invariant manifold theory, Nonlinear Anal. 20, no. 11, 1279-1302 (1993).
- [29] R. Johnson, X.B. Pan and Y.F. Yi, Positive solutions of super-critical elliptic equations and asymptotics Comm. Partial Differential Equations 18, no. 5-6, 977-1019 (1993).
- [30] C. Jones and T. Küpper, On the infinitely many solutions of a semilinear elliptic equation, SIAM J. Math. Anal. 17, 803-835 (1986).
- [31] Y. Li and W. Ni, Radial symmetry of positive solutions of nonlinear elliptic equations in  $\mathbb{R}^n$  Comm. Partial Differential Equations 18, no. 5-6, 1043-1054 (1993).
- [32] M. Marini and S. Matucci, A boundary value problem on the half-line for superlinear differential equations with changing sign weight, Rend. Istit. Mat. Univ. Trieste 44, 117-132 (2012).
- [33] S. Matucci, A new approach for solving nonlinear BVP's on the half-line for second order equations and applications Math. Bohem. 140, no. 2, 153-169 (2015).
- [34] J. D. Murray: Mathematical biology. II. Spatial models and biomedical applications. Third edition, Interdiscip. Appl. Math. 18, Springer, New York, (2003).
- [35] Ni W.M. and Serrin J., Nonexistence theorems for quasilinear partial differential equations, Rend. Circolo Mat. Palermo (Centenary supplement), Series II 8, 171-185 (1985).
- [36] D. Papini and F. Zanolin, Periodic points and chaotic-like dynamics of planar maps associated to nonlinear Hill's equations with indefinite weight., Georgian Math J. 9, 339-366 (2002).
- [37] S. I. Pohozaev, Eigenfunctions of the equations  $\Delta u + \lambda f(u) = 0$ , Soviet Math. Dokl 5, 1408-1411 (1965).
- [38] S. Terracini, On positive entire solutions to a class of equations with a singular coefficient and critical exponent, Adv. Differential Equations 1, 241-264 (1996).
- [39] X. Wang, On the Cauchy problem for reaction-diffusion equations, Trans. Am. Soc. 337, no. 2, 549-590 (1993).

- [40] E. Yanagida, Structure of radial solutions to  $\Delta u + K(|x|)|u|^{p-1}u = 0$  in  $\mathbb{R}^n$ , SIAM J. Math. Anal. 27, no. 3, 997-1014 (1996)